

GRAPH THEORY

①

1. GRAPH THEORY-1

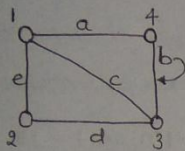
A Graph is represented as $G = (V, E)$

$V =$ Vertices set = $\{v_1, v_2, \dots, v_n\}$

$E =$ Edges set = $\{v_i \rightarrow v_j, \dots\}$

Now, $|V| =$ No. of vertices in the Graph = order of the Graph

$|E| =$ No. of edges = size of the Graph



$G = (V, E)$

$V = \{1, 2, 3, 4\}$

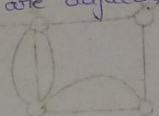
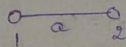
$E = \{a, b, c, d, e\}$

$a = \{1, 4\}$ $d = \{2, 3\}$

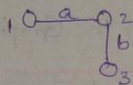
$b = \{4, 3\}$ $e = \{1, 2\}$

$c = \{1, 3\}$

$\Rightarrow$ Adjacent vertices:- Vertices having common edge are adjacent



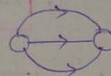
$\Rightarrow$ Adjacent edges:- If two vertices have an edge in common then the edges are called Adjacent edges.



$\Rightarrow$ Self-loop:-



$\Rightarrow$ Multiedge:-



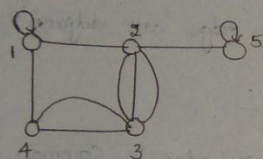
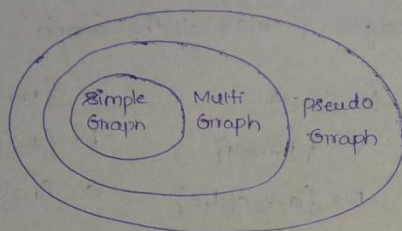
$\Rightarrow$ Simple Graph is the one which donot have self loops and multiedges.

Graph	Self loops	Multiedges
General/pseudo Graph	✓	✓
MultiGraph	✗	✓
Simple Graph	✗	✗

2. GRAPH THEORY- 2

Degree of the vertex: NO. of edges incident on it (a particular vertex) is called the degree of that vertex (Counting loops twice)

⇒ Every simple Graph is a multigraph as well as pseudo Graph. Similarly a multigraph is a pseudo Graph.



$$\begin{aligned} \deg(1) &= 4 & \deg(4) &= 3 \\ \deg(2) &= 5 & \deg(5) &= 3 \\ \deg(3) &= 5 \end{aligned}$$

$$\therefore \text{sum of the degrees} = 20.$$

$$\text{No. of edges in the Graph} = 10.$$

$$\therefore \text{sum of the degrees} = 2(\text{No. of edges}) \rightarrow \text{Hand Shaking Lemma}$$

$$\therefore \sum_{v \in V} \deg(v) = 2|E|$$

2. The no. of vertices with odd degree in a Graph is always even.

$$\sum \deg(v) = 2|E|$$

$$\sum_{\text{odd}} \deg(v) + \sum_{\text{even}} \deg(v) = \text{even}$$

$$\Rightarrow \sum_{\text{odd}} \deg(v) = \text{even} - \text{even} = \text{even}$$

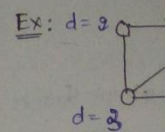
$$\therefore \sum_{\text{odd}} \deg(v) = \text{even}$$

∴ The no. of vertices with odd degree in a Graph is always even

3. GRAPH TH

Degree Sequence

(Descending)



⇒ Now, Give a deg se or not solution

HAVEL-H

⇒ put the

⇒ Remove

⇒ Subtract

⇒ Repeat

Stop this p

⇒ we get

⇒ If we ge

⇒ Not cho

Ex:

1) $(3, 2, 1, 1, 1)$

2)

3) $(6, 5, 4, 1, 1, 1, 1)$

⇒

Now

Sub

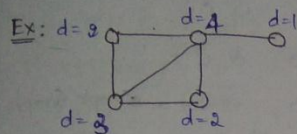
next

②

3. GRAPH THEORY-3

⑤

Degree Sequence: The Arrangement of degrees in Non-Ascending order (Descending order) is called the Degree Sequence.

Degree Sequence = $(4, 3, 3, 2, 1)$

→ Now, Given a graph finding the degree sequence is easy but given a deg sequence finding whether there exists atleast one simple Graph or not is difficult. we use "HAVEL-HAKIMI PROCEDURE" to find solution to such problem.

HAVEL-HAKIMI PROCEDURE

① put the degree sequence in Descending order

② Remove the highest degree (Let it be k)

③ Subtract one from the remaining k vertices

⇒ Repeat the steps ①②③

Stop this procedure when

⇒ we get all 0 entries ⇒ simple Graph exists

⇒ If we get Negative values ⇒ Simple Graph Not possible

⇒ Not enough edges ⇒ No Simple Graph.

Ex:

1) $(3, 2, 1, 1, 0)$ ⇒ sum of degrees should be even ⇒ here we get '7'

∴ No such Graph exists.

2)

$(3, 2, 1, 0, 0)$

$(1, 0, -1, 0)$

↳ Negative edge = No such Graph exists.

3) $(6, 5, 4, 3, 3, 1)$

⇒ $(5, 4, 3, 3, 1)$

Now cut '5' and

subtract '1' from the

next 5 nodes but there are only 5 ⇒ so No Simple Graph exists.

4) which of the following degree sequence doesnot correspond a Simple Graph

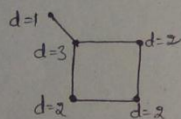
i) $(7, 6, 5, 4, 4, 3, 2, 1)$ $\Rightarrow$ ~~(7, 6, 5, 4, 4, 3, 2, 1)~~ (4)
~~(5, 4, 3, 3, 2, 1, 0)~~
~~(3, 2, 2, 1, 0, 0)~~
~~(1, 1, 0, 0, 0)~~
 $(0, 0, 0, 0) = \text{simple Graph exists}$

2) $(6, 6, 6, 6, 3, 3, 2, 2)$ $\Rightarrow$ ~~(6, 6, 6, 6, 3, 3, 2, 2)~~
~~(5, 5, 5, 2, 2, 1, 2)~~
~~(4, 4, 1, 1, 0, 2)~~
 $(3, 0, 0, -1, 1) \rightarrow (-ve) = \text{No simple Graph}$

4. GRAPH THEORY-4

Min degree = " δ " = Min of all the degrees of the particular Graph

Max degree = " Δ " = Max of all the degrees of the Graph is called Max deg.



$$\begin{matrix} \delta = 1 \\ \Delta = 3 \end{matrix}$$

$\Rightarrow$ If ' G ' is a graph with V - Vertices and e - Edges then,

$$\delta \leq \frac{2e}{V} \leq \Delta$$

Proof:

$$\sum \deg(v) = 2|E|$$

$\Rightarrow$ Replacing each degree with min. degree

$$\underbrace{\delta + \delta + \delta + \dots + \delta}_{V \text{ times}} \leq 2e$$

$\Rightarrow V \text{ times}$

$$\therefore \boxed{V \cdot \delta \leq 2e}$$

$\Rightarrow$ Replacing with max degree we get

$$= \Delta + \Delta + \Delta + \dots + \Delta \geq 2e$$

$$= \boxed{V \cdot \Delta \geq 2e} \Rightarrow \boxed{2e \leq V \cdot \Delta}$$

$$\therefore V \cdot \delta \leq$$

$$\Rightarrow \boxed{\delta \leq}$$

$\Rightarrow G$ is a Gra
of Vertices?

Now,

$\Rightarrow$

$\Rightarrow G$ is a G
edges in H

$$= 2e$$

$$= 2e$$

$$= e$$

5. GRAPH THE

Special Graphs

Null Graph:

Cycle Graph

The cycle Gra
edges of $v_1, v_2, \dots$

$$C_3 =$$

$\Rightarrow$ In cycle Gra

Simple Graph

(4)

$$\therefore V \cdot \delta \leq 2e \leq V \cdot \Delta$$

$$\Rightarrow \boxed{\delta \leq \frac{2e}{V} \leq \Delta}$$

$\Rightarrow G$ is a Graph with 11 edges and $\min \deg = 3$ then what is the max no. of vertices?

Simple Graph exists.

$$\text{Now, } \delta \leq \frac{2e}{V}$$

$$\Rightarrow 3 \leq \frac{2 \times 11}{V} \Rightarrow V \leq \frac{2 \times 11}{3}$$

$$\Rightarrow V \leq 7.33$$

$$\Rightarrow \boxed{V=7}$$

Simple Graph

$\Rightarrow G$ is a Graph with 12 vertices and $\max \text{ degree} = 4$, Then, max no. of edges in the Graph G is —?

$$= 2e \leq V \cdot \Delta$$

$$= 2e \leq 12 \times 4$$

$$= e \leq 24 \Rightarrow \boxed{e=24} \text{ (max no)}$$

Graph

called Max deg.

5. GRAPH THEORY 5

Special Graphs:

Null Graph: A graph with no edges and n -vertices is called Null Graph.

$$N_1 = 0$$

$$N_2 = 0 \quad 0$$

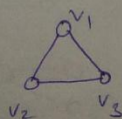
$$N_3 = \begin{matrix} 0 & 0 \\ & 0 \end{matrix}$$

$$N_4 = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$$

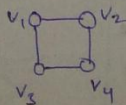
Cycle Graph

The cycle Graph is a simple Graph with n -vertices $\{v_1, v_2, \dots, v_n\}$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$

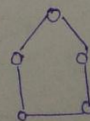
$C_3 =$



$C_4 =$

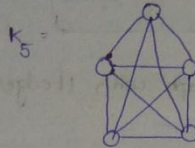
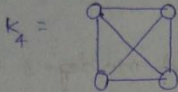
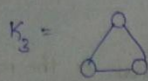


$C_5 =$



$\Rightarrow$ In cycle Graph C_n deg of each vertex will be '2'. | No. of vertices = edges = n |

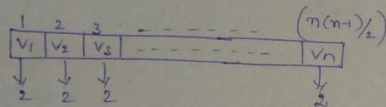
Complete Graph: complete Graph is a Simple Graph in which every pair of vertices are adjacent, Complete Graph denoted by K_n .



	Vertices	Edges	deg(v)
K_n	n	nC_2	$(n-1)$

$\Rightarrow$ Max no. of edges in simple Graph with 'n' edges = $nC_2 = \frac{n(n-1)}{2}$

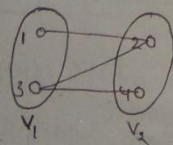
$\Rightarrow V = \{v_1, v_2, v_3, \dots, v_n\}$ of n -vertices then how many simple Graphs are possible?



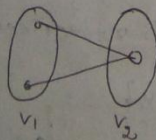
$\therefore$ Total possible no. of simple graphs = $2^{\frac{n(n-1)}{2}}$

Bipartite Graphs:

A Graph $G(V, E)$ is Bipartite Graph if the vertex set can be partitioned into two sets V_1 and V_2 such that every edge is in between vertex of V_1 and vertex of V_2 .



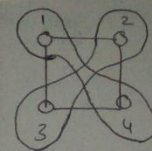
$\Rightarrow$ These should not be any edges between the vertices in a vertex set. (i.e. No edge between (1,3) and (2,4))



Bipartite



{ It is complete and cannot be Bipartite }



C_n is
 C_m is

5. MAX NO. OF

$\Rightarrow$ which of the

n -vertices

(a) n^2 (b)

$\Rightarrow$ Bipartite Gra

Now, let us ass

$\therefore$ To get th

7. COMPLETE

Complete Bipartite

Bipartite Gra

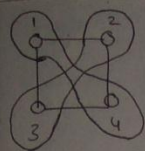
$K_{1,1} =$

$K_{1,2} =$

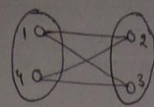
$K_{1,3} =$

every pair

(6)



$\equiv$



$\therefore$ The given graph is Bipartite Graph.

(7)

C_n is Bipartite if n = even | A complete Graph K_n can never be Bipartite Graph.
 C_n is not Bipartite if n = odd

5. MAX NO-OF EDGES IN A COMPLETE BIPARTITE GRAPH

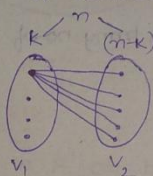
$\Rightarrow$ which of the following is the max no of edges in Bipartite graph with n -vertices

- (a) n^2 (b) $n^2/2$ (c) $n^2/4$ (d) $n/2$

$\Rightarrow$ Bipartite Graph with 6 vertices.

Graphs are possible?

Now, let us assume the ' n ' vertices are divided into k and $(n-k)$ vertices

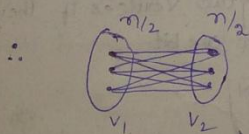


$\Rightarrow$ To get the max no of edges then every vertex in V_1 should be matched with all the vertex in V_2
 $\therefore$ The No. of edges = $k(n-k)$

$\therefore$ To get the max value of k , $\frac{d}{dk} (k(n-k)) = 0$

$$= (n-k) + k(-1) = 0$$

$$\Rightarrow k = n/2$$



$$\Rightarrow \text{No. of edges} = n/2 \times n/2 = n^2/4$$

7. COMPLETE BIPARTITE GRAPH AND REGULAR GRAPHS.

Complete Bipartite Graph: If we try to put every possible edge in the

Bipartite Graph then it is called Complete Bipartite Graph ($K_{m,n}$)

$$K_{1,1} = \text{---}$$

$$K_{1,2} = \text{---}$$

$$K_{1,3} = \text{---}$$

$$K_{2,1} = \text{---}$$

$$K_{2,2} = \text{---}$$

$$K_{2,3} = \text{---}$$

$$K_{3,1} = \text{---}$$

$$K_{3,2} = \text{---}$$

$$K_{2,3} = \text{---}$$

	v	e	d(v)
K_{mn}	$m+n$	$m*n$	$\begin{cases} m: \text{ve}_v \\ n: \text{ve}_v \end{cases}$

$$|V_1| = m$$

$$|V_2| = n$$

⑧

Regular Graph:

A Graph in which every vertex has same degree is called Regular graph.

⇒ Every complete Graph (K_n) is always a Regular Graph.

Ex: $N_n = 0$ (Null Graph) (0- Regular graph)

$C_n = 2$ (2- Regular Graph)

8. N- Cube

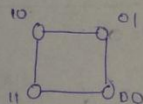
Now 1 cube means with 1 bits how many no. of vertices are possible = $2^1 = 2$

1-cube =

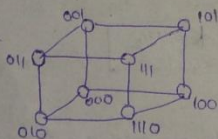


Now, 2 cube means with 2-bits how many no. of vertices are possible = $4 = 2^2$

2-cube =



Similarly 3-cube =



An edge is present between two vertices if they differ by one bit.

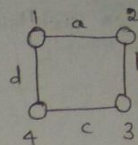
∴ No. of Vertices in N-cube Graph = (2^N)

Deg. of Each Vertex = N

No. of Edges = $N * 2^{N-1}$

9. SUB-GRAPH

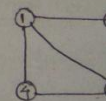
A Graph +



10. ADJACE

A Graph

⇒ No. of
⇒ Graph has
→ For Graph



11. ISOMOR

Isomorphism

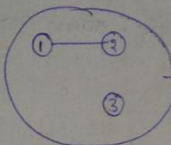
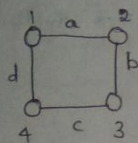
different

Ex:

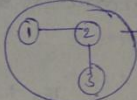
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9. SUB-GRAPHS

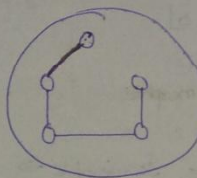
A Graph $H(V', E')$ is a subgraph of $G(V, E)$ if $V' \subseteq V$
 $E' \subseteq E$



→ Sub Graph of given Graph



→ Induced sub graph (If i take all the possible edges between the chosen vertices that are given in the Actual Graph).



→ Spanning sub Graph (If i take all the vertices of a Graph it is called Spanning sub Graph. It may contain edges also).

10. ADJACENCY MATRIX

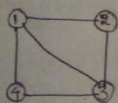
A Graph $G(V, E)$, $|V| = n$ it can be represented as $n \times n$ -matrix

$$A[G] = \begin{cases} 0 & \text{if } \{i, j\} \notin E \\ 1 & \text{if } \{i, j\} \in E \end{cases}$$

⇒ No provision for parallel edges.

⇒ Graph has no self loops iff the matrix diagonal entries are zeros.

→ For Graph having no self loops $\deg(v_i) = \text{sum of entries in the } i^{\text{th}} \text{ row}$.



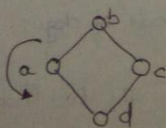
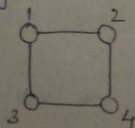
$$A[G] = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$\rightarrow \deg \text{ of } (1) = 3$ (No. of 1's in that row)
 $\rightarrow d(2) = 3$
 $\rightarrow d(3) = 3$
 $\rightarrow d(4) = 2$

11. ISOMORPHISM INTRODUCTION

Isomorphism: Isomorphic graphs are the graphs that are drawn differently.

Ex:



The time complexity for checking whether the Graphs are isomorphic or not is $O(n!)$

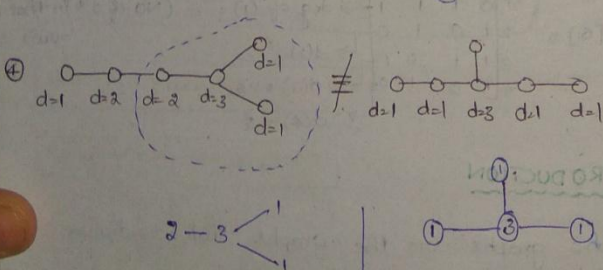
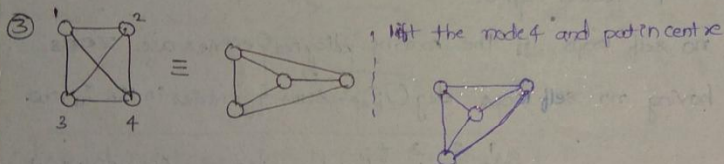
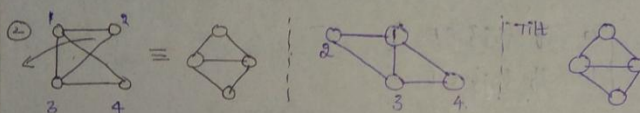
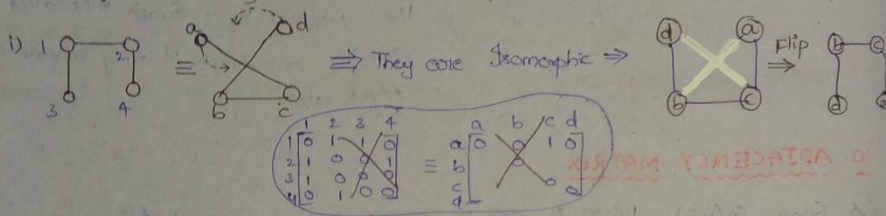
⇒ Isomorphic means two graphs should have same no. of vertices and their adjacencies should be preserved.

⇒ for the above examples, the adjacency matrix is same

$$\begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix} \equiv \begin{matrix} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

∴ The two graphs are isomorphic

Which of the following Graphs are Isomorphic



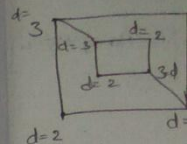
∴ The node having $\text{deg}=3$ is adjacent to nodes of $\text{deg}=2$ and $\text{deg}=1, \text{deg}=1$

here the node of $\text{deg}=3$ is not adjacent to the node of $\text{deg}=2$

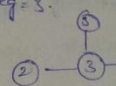
∴ The two Graphs are not isomorphic

12. ISOMORPH

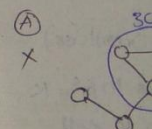
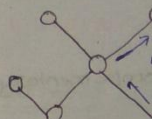
Are these



In this graph of $\text{deg}=3$ is nodes of $\text{deg}=3$.



⇒ which of



13. SELF

Complement

Let $G(V, E)$

such that

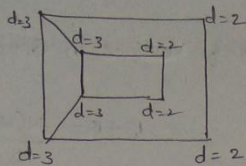
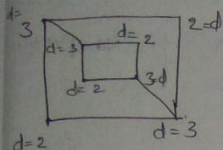
Ex

isomorphic or not

12. ISOMORPHISM EXAMPLES

(11)

Are these Graphs Isomorphic?



First check

1) No. of vertices and Edges

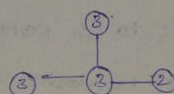
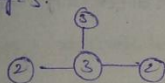
$$|V(G_1)| = |V(G_2)|$$

$$|E(G_1)| = |E(G_2)|$$

3) Check for Adjacency

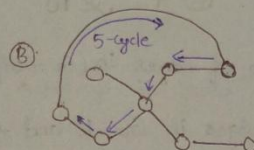
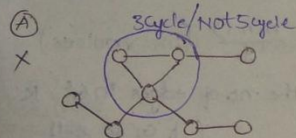
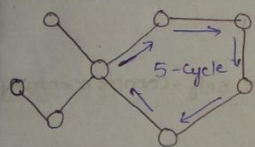
In this graph every node of deg=3 is adjacent to nodes of deg=2, deg=2, deg=3.

Every node of deg=3 is adjacent to nodes of deg=2, deg=3, deg=3.



∴ There is a conflict ∴ These Graphs are not Isomorphic

⇒ which of the graphs is Isomorphic to the Graph



And you can check degree Sequence also.

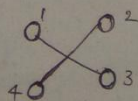
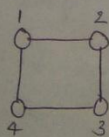
13. SELF COMPLEMENTING GRAPHS

Complement of a Graph

Let $G(V, E)$ be a simple Graph the complement of Graph $G = G^c = (V, E^c)$

Such that two vertices are adjacent in G^c if they are not adjacent in G .

Ex



No. of edges in G + No. of edges

$$\text{in } G^c = n_c$$

(∵ G, G^c forms Complete Graph)

⇒ If the no. of edges in $G = 13$ and No. of edges in $G^c = 15$ then the then what is the no. of vertices in G ? (12)

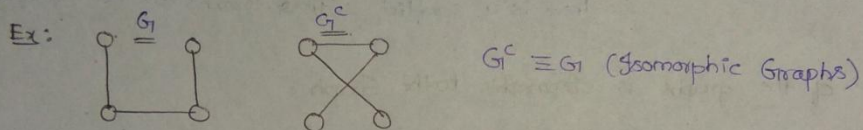
⇒ we know that the no. of edges in $G + G^c = \frac{N(N-1)}{2}$ [N = No. of vertices]

$$= 13 + 15 = \frac{N(N-1)}{2}$$

$$= \boxed{N = 8}$$

Self-complementing Graph

A graph which is isomorphic to its complement is called self-complementary graph. which means if G^c is isomorphic to G then it is called Self Complement graph.



⇒ which of the following cannot be the no. of vertices in self-complementary Graph. (A) 4 (B) 5 (C) 9 (D) 10

We know that the no. of edges in $(G + G^c) = \frac{n(n-1)}{2}$ (n = vertices)

⇒ Let the no. of edges in $G = K$ and so the no. of edges in $G^c = K$ (if G is self complementary)

$$\therefore 2K = \frac{n(n-1)}{2}$$

⇒ $K = \frac{n(n-1)}{4}$ | here K should be integer not fractions.

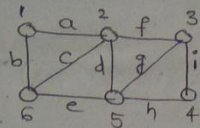
∴ n should be multiple of 4 or $(n-1)$ should be ÷ible by 4.

14. CONNECTED COMPONENTS

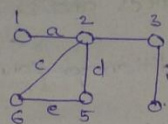
(15)

Edge Sequence: Sequence of edges starting and ending with vertex

Walk: Edge sequence in which no edge is repeated (vertices can be repeated but not the edges).



⇒ walk



(Vertex 2 is visited twice (vertex may be repeated but not the edge)).

closed walk: A walk in which start and end at same vertex is called closed walk otherwise it is open walk (vertex may be repeated).

path: A path is an open walk in which no vertex is repeated.

cycle: A cycle is a closed walk in which no other vertex is repeated.

Results:

1. A graph is Bipartite if every cycle in the graph is even cycle.

Connected: Two vertices are said to be connected if there exists at least one path between them.

⇒ The maximal connected subgraph is called "Component".

Results:

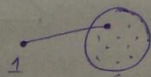
1) A simple Graph with n -vertices and k -components has at most $(n-k)(n-k+1)$ edges.

15. SUFFICIENT CONDITION FOR CONNECTEDNESS

The sufficient condition for connectedness is.

→ G is a simple Graph with n -vertices. the (no of edges) $> \frac{(n-1)(n-2)}{2}$

The ' G ' is connected.



$(n-1)$ vertices

$$(n-1)_2 \text{ edges} = \frac{(n-1)(n-2)}{2}$$

⇒ which of the following Simple graphs are always connected. (14)

- a) G_1 with 5 vertices and 5 edges
- b) G_1 with 6 vertices and 9 edges.
- c) G_1 with 7 vertices and 13 edges.
- d) G_1 with 8 vertices and 22 edges

⇒ The sufficient condition for connectedness is $(\text{no. of edges}) > \frac{(n-1)(n-2)}{2}$

$$(\text{No. of edges}) > \frac{(n-1)(n-2)}{2}$$

i) 5V, 5E $\Rightarrow 5 > \frac{(5-1)(5-2)}{2}$ } Disconnected Graph. (Not connected)
 $= \boxed{5 > 6}$ false

ii) 6V, 9E $\Rightarrow 9 > \frac{(6-1)(6-2)}{2}$ } Disconnected.
 $= \boxed{9 > 10}$ = false

iii) 7V, 13E $\Rightarrow 13 > \frac{(7-1)(7-2)}{2}$ } (iv) 8V, 22E } Connected
 $= \boxed{13 > 15}$ false $22 > \frac{7 \times 6}{2}$
 $\boxed{22 > 21}$

16. COMPLEMENTATION AND CONNECTEDNESS

Result: At least one of the Graphs G_1 or G_1^c is connected.

⇒ which of the following is always true?

- S_1 : G_1 is connected, then G_1^c is disconnected
- S_2 : G_1 is disconnected, then G_1^c is connected.

a) only S_1 is True (Not Necessarily true)

b) only S_2 is True

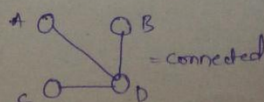
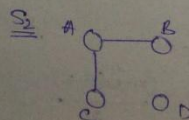
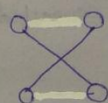
c) Both

d) None

S_1 : G_1



G_1^c



17. CUT VER

⇒ To make

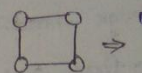
may stem

Removal of

Removal of

CUT EDGE @

the Graph



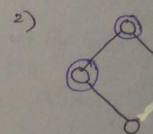
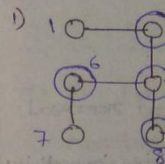
⇒ If there

⇒ If there

CUT VERTE

A single ve

cut vertex



⇒ A Graph

Biconne

17. CUT VERTEX AND CUT EDGE

⇒ To make a Graph disconnected we may remove some edges or we may remove some vertices.

Removal of an edge :



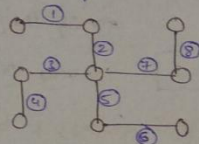
Removal of a vertex :



CUT EDGE (OR) BRIDGE : A single edge whose removal of it makes the Graph dis-connected is called cut edge.



⇒ No cut edges are present



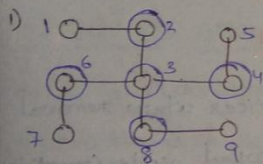
No. of cut edges = 7

⇒ If there is a tree every edge is a cut edge

⇒ If there is no cycle then every edge is a cut edge

CUT VERTEX AND ARTICULATION POINT

A single vertex whose removal disconnects the connected Graph is called cut vertex or "ARTICULATION POINT".

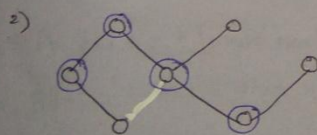


cut vertices / Articulation points are = {2, 3, 4, 8}

C = connected
D = Disconnected

possible cases

possible



No. of cut vertices are (a) 3 (b) 4 (c) 5 (d) 6

⇒ A Graph having no Articulation point / cut vertex is called a

Biconnected Graph

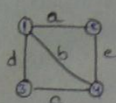
= All cycles are Bi-connected Graphs.



18. CUT SET AND EDGE CONNECTIVITY

Cut set: A set of edges whose removal disconnects the graph is called cutset. There can be more cutset for a given graph.

Ex:



$$C_1 = \{a, b, c, d, e\}$$

$$C_4 = \{a, b, c\}$$

$$C_2 = \{a, b, d\}$$

$$C_5 = \{c, d\}$$

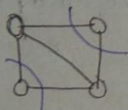
$$C_3 = \{a, e\}$$

Minimum cut set

Edge-connectivity (λ):

The min. no. of edges whose removal disconnects the graph is called

Edge-connectivity (λ).



$$\lambda = 2$$

$\Rightarrow$ If the Graph contains cycle then edge connectivity λ will always be greater than one

$$\lambda > 1$$

$\Rightarrow$ The edge connectivity is always upper bounded by min degree $\therefore \lambda \leq \delta$

W.K.T

$$\lambda \leq \delta \leq \frac{2E}{V}$$

$$\lambda \leq \frac{2E}{V}$$

19. VERTEX CONNECTIVITY AND EDGE CONNECTIVITY

Vertex Connectivity ($K = \kappa$): The min no. of vertices whose removal disconnects the graph (or) leaves trivial graph is called vertex connectivity.

$0 \rightarrow$ Trivial Graph (K_1).

$$K \leq \lambda \leq \delta \leq \frac{2E}{V}$$

G	λ	K
C_n	2	2
W_n	3	3
K_n	$(n-1)$	$(n-1)$
$K_{m,n}$	$\min(m, n)$	$\min(m, n)$

wheel Graph, W_n contains $(n+1)$ vertices

W_n has $2(n-1)$ edges

deg of each node = n (which are in cycle)

And vertex degree = $n-1$

20. WH

In any

$\Rightarrow$ A G

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21. EULE

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Graph

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Result: A

20. WHITNEY'S THEOREM

(17)

Graph is

In any Graph

$$K \leq \gamma \leq \delta$$

$$\text{Vertex Connectivity} \leq \text{Edge Connectivity} \leq \text{Mindegree}$$

⇒ A Graph with 11 edges and min degree is 4. what is the max value of vertex connectivity?

Acc. to Whitney theorem $K \leq \gamma \leq \delta \leq \frac{2E}{V}$

$$\Rightarrow \begin{array}{|c|c|} \hline K \leq \frac{2E}{V} & \delta \leq \frac{2E}{V} \\ \hline \end{array} \Rightarrow \begin{array}{|c|} \hline K \leq 4 \\ \hline \end{array}$$

∴ max value $K=4$

⇒ A Graph 'G' with 11 edges and 7 vertices is given what is the max value of K (vertex connectivity)?

$$\frac{2E}{V} \Rightarrow K \leq \frac{2E}{V}$$

$$K \leq \frac{2 \times 11}{7}$$

$$K \leq 3.1 \Rightarrow \text{max. value} = 3 \Rightarrow K=3$$

21. EULER'S GRAPH

⇒ If you can start from a vertex and if you can travel all edges exactly once and again come back to the same vertex then such a walk is called Euler circuit and such a Graph is called Euler Graph.

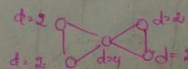
⇒ If you start from a node and come back to the same node by covering all the vertices then it is called Eulerian circuit / Euler path

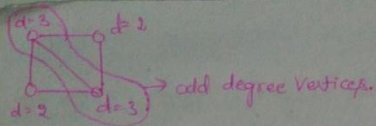
⇒ If you start from a node and cover all the nodes and reach some other node which is final then it is called unicursal path and the Graph is called unicursal Graph

⇒ If a graph has to be Euler Graph it should contain Even degree.

(The vertices should have Even degree.)

Result: A multigraph is Euler Graph iff degree of every vertex is even.

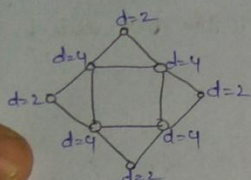




⇒ Not Euler graph. So unicursal Graph.

(18)

⇒ Is this Graph a Euler Graph



All degrees are Even $\therefore$ The Graph is Euler Graph.

Result: A Multigraph is unicursal graph if there are exactly 2 vertices of odd degree.

22. HAMILTONIAN GRAPH (H-Graph)

→ In Euler Graph every edge should be covered and vertex can be repeated

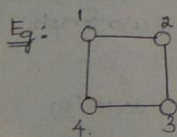
→ In "Hamiltonian Graph" all the vertices should be covered and nothing should be repeated (Need not cover all the edges)

E-E (Euler Graph - should cover all the Edges).

⇒ You have to draw a cycle in the Graph such that it covers all the vertices

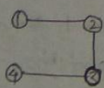
⇒ A Graph containing Hamiltonian cycle is called Hamiltonian-Graph.

⇒ A path containing all the vertices of the Graph is called "Hamiltonian Path".

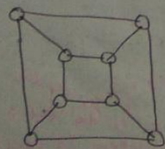


1-2-3-4-1 = Hamiltonian cycle

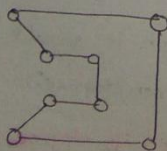
= Hamiltonian Graph (H-Graph)



= 1-2-3-4 = Hamiltonian path (Remove one edge from Hamiltonian-cycle then you get H-path).



=



= Hamiltonian cycle = Hamiltonian path

Results:

1) If G is

i) Hami

2) Ha

3) H-

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Graph

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Ex:



⇒ An Rep

Graphs.

(18)

Results:

i) If 'G' is a Hamiltonian Graph with n-vertices

1) Hamiltonian cycle, H-path contains n-vertices

2) Hamiltonian cycle contains n-edges.

3) H-path contains (n-1) edges.

⇒ In a Hamiltonian cycle each vertex should have (deg=2).

The sufficient condition for Hamiltonian cycle is (Not Necessary)

a) Dirac's theorem: (Simple Graph)

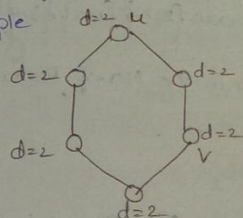
If min degree $\delta \geq n/2$ then G is Hamiltonian Graph

b) Ores Theorem: If $\deg(u) + \deg(v) \geq n$ (for $n \geq 3$) then for every pair of non-adjacent vertices, then 'G' is H-Graph.

⇒ If a Graph satisfies Dirac, Ores theorem G is Hamiltonian but if a

Graph doesn't satisfy above Theorems we cannot say 'G' is Non-H-Graph

For example



Here $\delta = 2$ $n = 6$

$$\delta \geq n/2 \Rightarrow 2 \geq 3 \Rightarrow (2 \geq 3) \text{ false}$$

but the Graph is Hamiltonian.

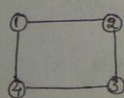
$\deg(u) + \deg(v) = 4$ ($4 \geq 6$) false but G is H-graph.

Path:

23. POWERS OF ADJACENCY MATRIX

Let $A[G]$ be the Adjacency matrix. then $[A(G)]^n$ Represents power of Adjacency matrix.

Ex:



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

→ Represents the no. of paths of length 1 from 1 to 2

→ Represents the no. of paths of length 2 from 1 to 3

⇒ A^n Represents the no. of paths b/w two vertices of length 'n'.

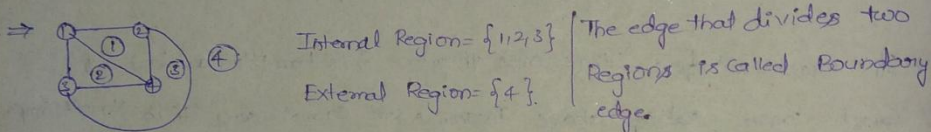
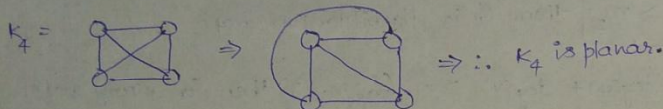
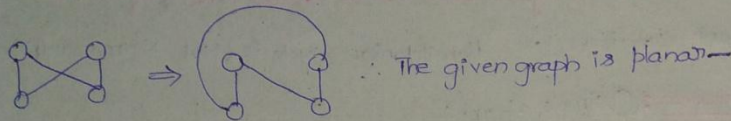
24. PLANAR GRAPHS

⇒ If we can draw a Graph on a paper without crossovers then the Graph is planar. This concept is used in VLSI design.

⇒ Cross over:



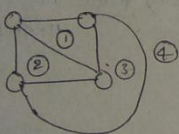
Ex:



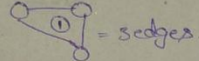
Here the Boundary edges are {2, 4}, {1, 4}, {3, 4}

Degree of the Region: Degree of the Region is the No. of Boundary edges touching it.

Ex:



Region	deg(R)
1	3
2	3
3	3
4	3



⇒ Sum of degrees of the Regions = Twice the no. of Boundary edges.

25. EULER FORMULA FOR SIMPLE GRAPH

Let 'G' be a connected planar graph with

V = vertices

E = edges

R = regions then

$$V - E + R = 2$$



$V = 4$
 $E = 5$
 $R = 3$

$$V - E + R = 4 - 5 + 3 = 2$$

⇒ Min de

⇒ Now,

Now, $V - E$

⇒ $V -$

⇒ 2

26. K_5 is

$K_5 =$



⇒ Since

27. $K_{3,3}$

W.K.T.

W.K.T.

⇒ $N - e$

(20)

⇒ Min degree of a Region in simple Graph = 3.

(21)

Now, $\sum \deg(r) = 2e$

$$= r_1 + r_2 + r_3 + \dots + r_n = 2e$$

$$= 3 + 4 + 5 + \dots + n = 2e$$

$$= 3 + 3 + 3 + \dots + 3 \leq 2e$$

$$= 3|r| \leq 2e \Rightarrow \boxed{3r \leq 2e}$$

Now, Euler formula

$$\Rightarrow V - e + r = 2$$

$$\Rightarrow V - e + \frac{2e}{3} \geq 2$$

$$\Rightarrow \boxed{e \leq 3V - 6}$$

$$\begin{aligned} 3r &\leq 2e \\ \Rightarrow r &\leq \frac{2e}{3} \end{aligned}$$

$$\Rightarrow \boxed{r \leq \frac{2e}{3}} \quad (\text{or}) \quad \boxed{e \geq \frac{3r}{2}}$$

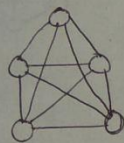
Now, $V - e + r = 2$

$$\Rightarrow V - \frac{3r}{2} + r \leq 2$$

$$\Rightarrow \boxed{2V - r \leq 4}$$

26. K_5 IS NOT PLANAR

$K_5 =$



K_5 = Kuratowski's Graph = $K_{3,2}$ is also called Kuratowski's Graph.

$$V = 5$$

$$E = 10$$

By Euler formula

$$V - e + r = 5 - 10 + r = 2$$

$$\Rightarrow \boxed{r = 7}$$

Now, $3r \leq 2e$

$$\Rightarrow 21 \leq 2 \times 10$$

$$= \boxed{21 \leq 20} \text{ false } \therefore K_5 \text{ is Not planar / It must not be connected}$$

⇒ Since K_5 is connected, K_5 is Not-planar Graph.

27. $K_{3,3}$ IS NOT PLANAR PART 1 AND PART 2

W.K.T. $V - e + r = 2$, Now let us assume that the min. deg = k then.

W.K.T. sum of degree = $2e$ and $V - e + r = C + 1$

$$= k + k + k + \dots + k = 2e$$

$$= k|r| \leq 2e \Rightarrow \boxed{k r \leq 2e}$$

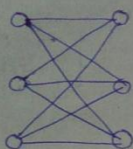
$$\Rightarrow \boxed{r \leq \frac{2e}{k}}$$

$$\Rightarrow V - e + \frac{2e}{k} \geq 2 \Rightarrow \boxed{e \leq \frac{k(V-2)}{k-2}} \quad (\text{or}) \quad \boxed{V - \frac{k r}{2} + r \leq 2}$$

↳ No. of connected components.

Now, let us check if $K_{3,3}$ is planar or not

$K_{3,3}$:



Min, deg of any region = 4

Now, $V - E + R = 2$

$$\Rightarrow 6 - 9 + R = 2$$

$$R = 5$$

Now $E \leq \frac{4(R)}{2}$

$$9 \leq 8 \text{ = False}$$

$\therefore K_{3,3}$ is not planar



Note: There won't be any Region of 'deg's' in Bipartite Graph.

$\therefore$ The Graphs that contain $K_{3,3}$ and K_5 as subgraphs are not planar.

* The Graph which is Non-planar with min. no. of edges is K_5 $\therefore$ The

Graphs with 2,3,4 vertices are planar. (Simple Graphs)

2) $K_{3,3}$ is Non-planar with min. no. of edges $\therefore$ Any simple Graph with edges < 8 is planar.

so, K_m is planar iff $m \leq 4$

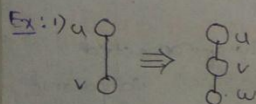
$K_{m,n}$ is planar iff $m \leq 2$ and $n \leq 2$



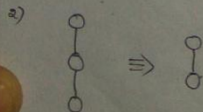
29. CHECKING FOR PLANARITY

Homomorphic operations

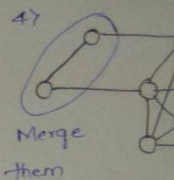
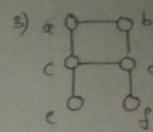
$\Rightarrow$ when ever you take a Graph and if we apply Homomorphic operations and we get another Graph then we can say that the two graphs are Homomorphic to each other.



$\Rightarrow$ Insertion of vertex of $d=2$ in series.



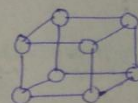
$\Rightarrow$ Removing a vertex



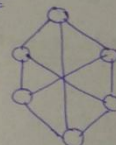
Kuratowski's

- A connected

Ex:



Ex:



Ex:

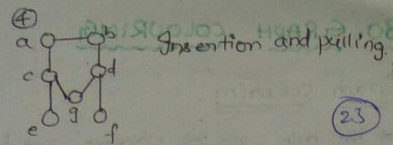
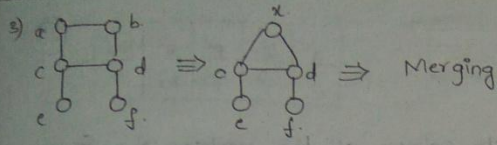
1) $V = 20$, de

what are

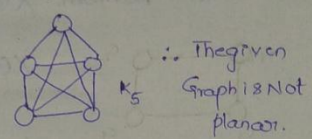
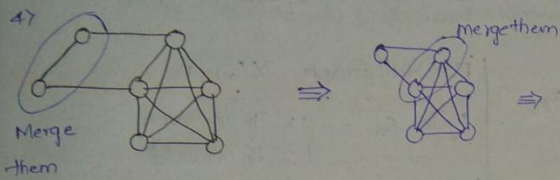
2d $V = 20$

$d(v_i) = 3$

Region (22)



(23)

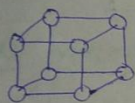


$\therefore$ The given Graph is Not planar.

Kuratowski's Theorem

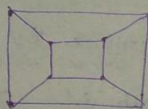
A connected graph is planar iff it is not homomorphic to K_5 or $K_{3,3}$.

Ex:



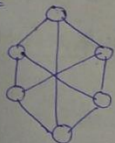
whether the Graph is homomorphic or not?

$\Rightarrow$

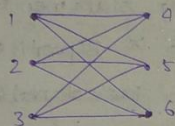


= planar (No cross over)

Ex:



$\Rightarrow$



= $K_{3,3}$ $\therefore$ Not planar Graphs.

Ex:

1) $V=20$, $\deg(v_i)=3$ for all $v_i \in V$ then sum of degrees = $2e = 60$ then

what are the no. of Regions.

8d $V=20$
 $d(v_i)=3$

sum of degree $\Rightarrow 2e = 60$

$\Rightarrow \boxed{e=30}$

Now, $V - e + r = 2$

= $20 - 30 + r = 2$

= $\boxed{r=12}$

at min case each region will have min deg of 3

$\therefore 3+3+3 \dots 3$ (12 times)

= 60

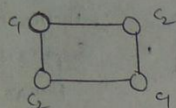
= $\boxed{2e}$

each vertex degree

30. GRAPH COLOURING

Graph colouring:

⇒ The min no. of colours reqd to colour all the vertices of a Graph such that no two adjacent vertices have same color is called chromatic number. $\chi(G)$.



$$\chi(G) = 2$$

For Null Graph $\chi(G) = 1$

$$\text{Cycle } C_n = \begin{cases} 2 & \text{if } n = \text{even} \\ 3 & \text{if } n = \text{odd} \end{cases}$$

$$\text{Wheel } W_n = \begin{cases} 3 & \text{if } n = \text{even} \\ 4 & \text{if } n = \text{odd} \end{cases}$$

$$\Rightarrow \chi_n = \{n\}$$

$$\Rightarrow \chi_{\min} = 2$$

Any Bipartite Graph, $\chi(G) = 2$

Properties

* 'G' is a graph with n-vertices then $\chi(G) \leq n$

* G_n is a sub-graph of Graph 'G' $\chi(G) \geq n$

* $\chi(G) \leq 1 + \Delta \rightarrow \text{max degree}$

* $\chi(G) = \frac{|V|}{|V| - \delta}$ | The following statements are Equivalent

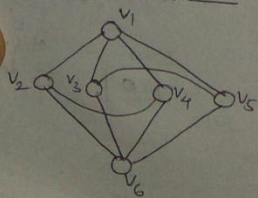
- 1> 'G' is Bipartite
- 2> 'G' is 2-colourable
- 3> Every cycle in 'G' is Evencycle.

31. GRAPH COLOURING EXAMPLE-1

There is no Algo which can find the chromatic no. of a graph. (Np-complete)

But we use some Greedy methods (Algo) like

Welsh-Powell Algorithm



$$\begin{aligned} d(v_1) &= 4 & d(v_2) &= d(v_3) = d(v_4) = d(v_5) = 3 \\ d(v_6) &= 4 \end{aligned}$$

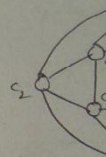
Now, sort the vertices in descending order of the degrees. = $v_1, v_6, v_2, v_3, v_4, v_5$

(24)

V_1	V_6	V_5
c_1	c_1	c_1

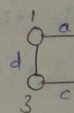
⇒ Start for
are n

32. GRAPH



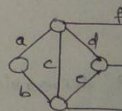
33. MATCH

Matching: S



Matching no:

Ex:



Edge cover

what are



24

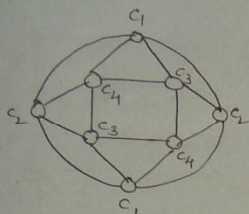
V_1	V_2	V_3	V_4	V_5
C_1	C_1	C_2	C_2	C_3

$$\chi(G) = 3$$

Graph

$\Rightarrow$ Start from ' V_1 '. Now apply 1 color to V_1 and the vertices that are not adjacent to it, and Repeat the procedure.

32. GRAPH-COLOURING Example-2

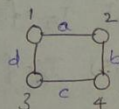


$$\chi(G) = 4$$

$\Rightarrow$ The Graph contains ' K_4 ' so the value of $\chi(G) \geq 3$.

33. MATCHING AND EDGE COVER

Matching: Set of Non-Adjacent edges.

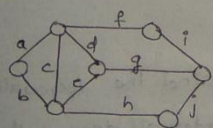


$$M_1 = \{a, c\} \quad M_2 = \{b, d\}$$

Matching no: $\alpha'(G)$: Max no. of Non-Adjacent edges. For above Graph

$$\alpha'(G) = 2$$

Ex:



$$M_1 = \{a, i, h\}$$

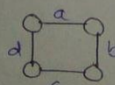
$$M_2 = \{f, g, b\}$$

$$M_3 = \{i, e, a\}$$

$$\alpha'(G) = 3$$

Edge cover:

What are all the edges that you should choose that covers all the vertices (Not isolated vertex)

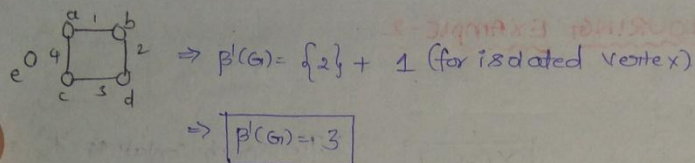
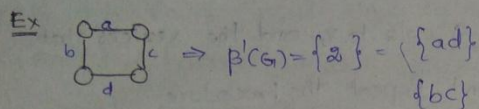


$$\Rightarrow \text{Edge cover} = \{abc\} \rightarrow \{abd\}$$

$$= \{bcd\} \rightarrow \{ac\}$$

$$= \{adc\} \rightarrow \{db\} \Rightarrow a, b$$

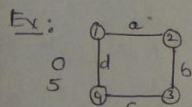
Edge covering NO: Min no. of edges that are required to cover all the vertices is called Edge covering NO. $[P'(G)]$



For any Graph $\alpha'(G) + P'(G) = n$

34. VERTEX COVER AND INDEPENDENT SET

Independent set: set of Non-Adjacent vertices is called Independent set



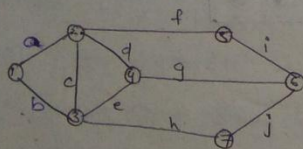
Independent set $I_1 = \{1, 3\}$ $I_2 = \{1, 4\}$ $I_3 = \{1, 5\}$ $I_4 = \{2, 4\}$ $I_5 = \{2, 5\}$ $I_6 = \{3, 5\}$ $I_7 = \{4, 5\}$ $I_8 = \{1, 3, 5\}$

Independence no: $\alpha(G)$

Max No. of elements in the largest Independent set is called Max NO.

For the above graph $\alpha(G) = 3$ $[I_8, I_7]$

Ex:



$I_1 = \{1, 4, 5, 7\}$

$I_2 = \{2, 6\}$

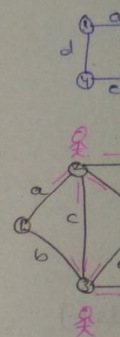
{ Find the decreasing order of degree of vertices }

or
{ Increasing order of vertices }

VERTEX COVER

Ex:

Vertex all the



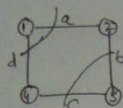
over

(26)

VERTEX COVER : Set of vertices that cover all the edges is called vertex cover.

(27)

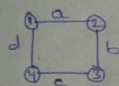
Ex:



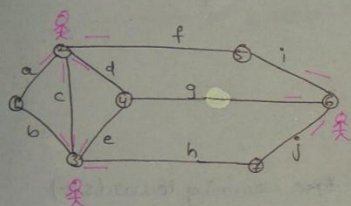
$$V_1 = \{1, 2, 3, 4\} \quad V_3 = \{2, 4\}$$

$$V_2 = \{1, 3\}$$

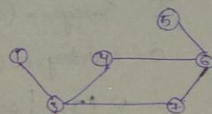
Vertex covering number: The min no. of vertices which can cover all the edges



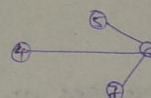
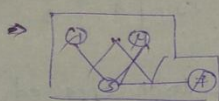
$$\beta(G) = 2$$



⇒ To get the vertex covering number remove the vertex with highest degree = {2, 3} so we can remove anyone.



⇒ Now remove the vertex with next highest degree = 5 (in main graph not the graph)



⇒ Now Remove = 6 then all vertices get removed.

∴ we have removed three vertices ∴ $\beta(G) = 3$

For any Graph, "G"

$$\alpha(G) + \beta(G) = n$$

$$\text{Independence no} + \text{Vertex covering no} = n$$

$$\text{Matching no} + \text{Edge covering no} = n$$

$$\alpha(G) + \beta(G) = n$$

35. PERFECT MATCHING

A matching which can cover all the vertices of a Graph

Ex:  $M_1 = \{a, c\}$ can cover all vertices

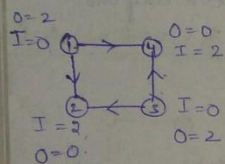
→ A graph G has perfect matching only if G has even no. of vertices.

* No. of perfect matchings in $K_{2n} = \frac{(2n)!}{2^n (n!)}$

⇒ Find the no. of perfect matchings in $K_3 \Rightarrow n=3 \Rightarrow \frac{6!}{2^3 3!} = \underline{15}$

⇒ No. of perfect matching in $K_n = n!$

36. DIRECTED GRAPH



$I =$ In-degree (No. of edges coming towards it)

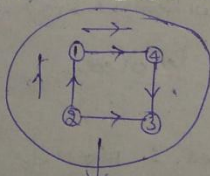
$O =$ out-degree (No. of edges outgoing from it)

v	In	out
1	0	2
2	2	0
3	0	2
4	2	0
Sum	4	4

In any Graph sum of

In-degrees = sum of out-degrees
= No. of Edges

⇒ Two vertices are said to be connected if there exists a "directed path" between them. In the below Graph ② and ④ are connected.

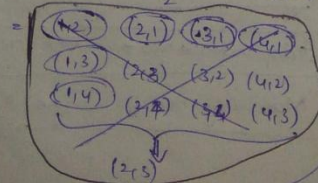


→ This Graph is unilaterally connected because between every pair there exists a directed path.

possible pairs

$$= {}^4C_2 = 6$$

- (1,2)
- (1,3)
- (1,4)
- (2,4)
- (2,3)
- (3,4)



(1,2)
(1,3)
(1,4)

(2,1) (1,4)
(2,4) (1,3)
(2,3) (3,4)