

SET THEORY AND ALGEBRA

SETS

1. SETS AND SUBSETS

Set: well-defined unordered collection of distinct elements

Ex: $A = \{1, 2, 3, 4\}$

$S = \{\text{set of all students in a class}\}$

Null set: Set with No elements is called Null set, denoted as ϕ or $\{\}$

subset: If every ele of A is also an element of 'B' then A is subset of B. ~~subset~~

$A = \{1, 2, 3\}$ $B = \{1, 2\}$ here $(B \subseteq A)$

Note: For every set 'A', 'A' and ' ϕ ' are called Trivial subsets of 'A'.

Proper subset: Any subset of 'A' which is not a trivial subset is called proper subset of 'A'.

Note: If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

2. POWER SET

Denoted by $P(A)$

$\Rightarrow$ If 'A' is finite set then set of all finite subsets of 'A' is called power set of 'A'. It is denoted by $P(A)$.

Ex: $A = \{a, b\}$

subsets of A = $\phi, \{a\}, \{b\}, \{a, b\}$

$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$

$\Rightarrow$ If a set contains 'n' elements ($|A| = n$) then $|P(A)| = 2^n$ elements.
contains

3. Compliment AND Difference

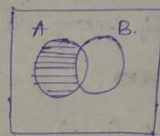
Universal set: set of all objects under discussion. denoted by 'U'

Compliment of a set: If 'A' is any set, then complement of 'A' denoted by $\bar{A}$ or A^c is called

$$A^c = \{x/x \notin A \text{ and } x \in U\}$$

Difference If 'A' and 'B' are two sets then

$$A - B = \{x/x \in A \text{ and } x \notin B\}$$



$$A = \{1, 2, 3, 4\}$$

$$B = \{3, 4, 5, 6\}$$

$$A - B = \{1, 2\}$$

4. UNION, INTERSECTION AND SYMMETRICAL DIFFERENCE

Set Intersection: $A \cap B = \{x/x \in A \text{ and } x \in B\}$

Set union: $A \cup B = \{x/x \in A \text{ OR } x \in B\}$

Note: If $A \cap B = \emptyset$, then 'A' and 'B' are disjoint sets.

Symmetric difference or Boolean sum: $A \Delta B = \{x/x \in A \text{ or } x \in B \text{ but } x \notin A \cap B\}$

$$\begin{aligned} A \Delta B &= (A - B) \cup (B - A) \\ A \Delta B &= (A \cup B) - (A \cap B) \end{aligned}$$

5. LAWS OF SETS

Commutative laws: (i) $A \cup B = B \cup A$
 $A \cap B = B \cap A$
 $A \oplus B = B \oplus A$

Associative laws:

$$1) (A \cup B) \cup C = A \cup (B \cup C)$$

$$2) (A \cap B) \cap C = A \cap (B \cap C)$$

$$3) (A \oplus B) \oplus C = A \oplus (B \oplus C)$$

Distributive law

$$1) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$2) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Modular laws

$$1) (A \cup B) \cap C = A \cup (B \cap C)$$

$$2) (A \cap B) \cup C = A \cap (B \cup C)$$

$$3) A \cup \emptyset = A$$

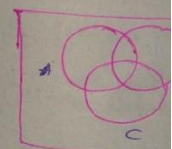
$$4) A \cap \emptyset = \emptyset$$

6. Example:

Which of the

$$a) A - (A - B) = B$$

$$b) A - (A - B) = \emptyset$$



Use this diagram for 3 sets

$$iii) A - (A - B) = B$$

$$= \{2, 3\} - \{2\} = \{3\}$$

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Distributive laws:

$$1) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$2) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Modular laws

$$1) (A \cup B) \cap C = A \cup (B \cap C) \text{ iff } A \subseteq C$$

$$2) (A \cap B) \cup C = A \cap (B \cup C) \text{ iff } C \subseteq A$$

$$3) A \cup \phi = A \quad 5) A \cup U = U, A \cap U = A$$

$$4) A \cap \phi = \phi \quad 6) A \cup A^c = U, A \cap A^c = \phi$$

Demorgan law

$$1) (A \cup B)^c = A^c \cap B^c$$

$$2) (A \cap B)^c = A^c \cup B^c$$

$$3) A - (B \cup C) = (A - B) \cap (A - C)$$

$$4) A - (B \cap C) = (A - B) \cup (A - C)$$

Idempotent law

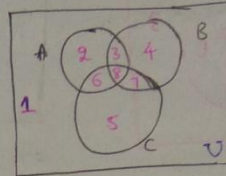
$$1) A \cup A = A$$

$$2) B \cap B = B$$

Absorption law

$$1) A \cup (A \cap B) = A$$

$$2) A \cap (A \cup B) = A$$



$$A \cup (B \cap C) = \{2, 3, 6, 8\} \cup \{7, 8\} \\ = \{2, 3, 6, 7, 8\}$$

$$(A \cup B) \cap (A \cup C) = \{2, 3, 4, 6, 7, 8\} \cap \{2, 3, 6, 8, 5\} \\ = \{2, 3, 6, 8, 7\}$$

6. EXAMPLE 1

Which of the following is not TRUE?

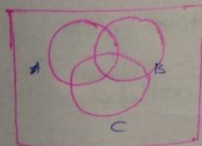
a) $A - (A - B) = B$

b) $A - (A - B) = (A \cap B)$

c) $(A \cap B) \cup (A \cap B^c) = A$

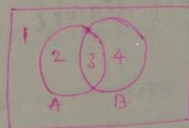
d) $B \cap (A \cup B) = A$

No. of Regions = 2^n
 n = no. of sets.



Use this diagram for

3 sets



for two sets.

i) $A - \{2, 3\} - \{4\} = \{3\}$

$\times B = \{3, 4\}$

ii) $(A \cap B) \cup (A \cap B^c) = A$

$= \{3\} \cup \{2, 3\} \cap \{1, 2\}$

$= \{3\} \cup \{2\} = \{2, 3\}$

$\neq A$

iii) $A - (A - B) = (A \cap B)$

$= \{2, 3\} - \{2\} = \{3\} = A \cap B$

iv) $B \cap (A \cup B) = A$

$= \{3, 4\} \cap \{2, 3, 4\}$

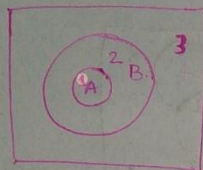
$= \{3, 4\} \neq A$

$= \text{False}$

1. Example-2

Which of the following is not True?

- a) If $A \subset B$, then $B^c \subset A^c$ ✓ (c) $A \cap p(A) = \emptyset$ ✓
 b) $A \cap p(A) = A$ ✗ (d) $p(A) \cap p(p(A)) = \{\emptyset\}$ ✓



$$\begin{aligned} 1) B^c &= \{3\} \\ A^c &= \{2, 3\} \end{aligned} \Rightarrow B^c \subset A^c \text{ (TRUE)}$$

$$2) A \cap p(A) = \emptyset$$

These are elements $\leftarrow A = \{a, b\}$

These are sets $\leftarrow p(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

$$A \cap p(A) = \{\emptyset\}$$

$$A \cap p(A) = \emptyset$$

$$3) p(A) \cap p(p(A)) = \{\emptyset\}$$

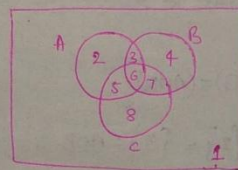
$$p(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

$$p(p(A)) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\{b\}\}, \{\{a, b\}\}, \{\emptyset, \{a\}\}, \{\emptyset, \{b\}\}, \{\emptyset, \{a, b\}\}\}$$

8. Example-3

Which of the following is NOT TRUE?

- a) $(A-B)-C = (A-C)-B$ ✓ = TRUE
 b) $(A-B)-C = (A-C)-(B-C)$ ✓ = TRUE
 c) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$ ✗ = FALSE
 d) $A - (B \cup C) = (A-B) \cap (A-C)$ ✓ = TRUE



$$2) A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$$

$$\{2, 3, 5, 6\} \oplus \{3, 4, 6, 7, 8\}$$

$$= \{2\} \cup \{2, 4, 7, 8\}$$

present here but not here () and present here but not here

$$\text{Now, } (A \oplus B) = (2, 5, 4, 7) \cup (2, 3, 7, 8) = (2, 3, 4, 7, 8) \neq \{2, 4, 7, 8\}$$

$$1) (A-B) = \{2, 3, 5, 6\} - \{3, 4, 6, 7, 8\} = \{2, 5\} - \{6, 7, 8\} = \{2\}$$

$$(A-C)-B = \{2, 5\} - \{3, 6, 7, 8\} = \{2\}$$

2. RELATIO

1. INTRO DI

Cartesian Pro

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

$\Rightarrow$ Cartesian

$\Rightarrow$ Here $(A \times B)$

$\Rightarrow$ In general

$\Rightarrow$ we take a

Relation:

The Relation

$$\Rightarrow \text{If } |A| = m$$

(Since Rel

contains

over the

be $2^m \times n$

Ex-1 Let R :

Ex-2: Let R :

Ex-3: Let A :

$$R: \{(x, y) \mid x \in A, y \in B\}$$

2. RELATIONS

1. INTRODUCTION TO RELATIONS

Cartesian Product: $A = \{1, 2, 3\}$ $B = \{a, b\}$

$$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

↳ Each ele is called Ordered pair.

⇒ Cartesian product = Cross product

⇒ Here $|A \times B| = 3 \times 2 = 6$.

⇒ In general if $|A| = m$, $|B| = n$ then $(A \times B)$ contains $(m \times n)$ Ordered pairs

⇒ we take a Relation from Cartesian product.

Relation:

The Relation is a subset of Cartesian product

for above example $R = \{(1, a), (1, b)\}$

$$1Ra \Rightarrow (1, a) \in R$$

$$R' = \{(1, a), (2, a), (3, a)\}$$

$$R'' = \{1Ra, 2Ra, 3Ra\}$$

⇒ If $|A| = m$ and $|B| = n$ then no. of Relations that can be formed = $2^{m \times n}$

(Since Relation is subset of Cartesian product, and Cross product contains $(m \times n)$ elements. Now, the no. of subsets that are possible over the Cartesian product set is $2^{m \times n}$, so the no. of relations will be $2^{m \times n}$.)

Ex-1: Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x < y\} = \{(1, 2), (2, 3), (4, 5), \dots\}$

Ex-2: Let $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : (x+y) \text{ is even}\} = \{(1, 3), (1, 5), (2, 4), \dots\}$

Ex-3: Let $A = \{1, 2, 3, 4\}$ $B = \{1, 2\}$

$R = \{(x, y) \in A \times B : x + y = 3\} = \{(1, 2), (2, 1)\}$

2. REFLEXIVE RELATION

Reflexive Relation: A Relation 'R' on set 'A' is said to be Reflexive

if $(xRx) \forall x \in A$

$$A = \{1, 2, 3\}$$

$$A \times A = \{(1,1) (1,2) (1,3) \\ (2,1) (2,2) (2,3) \\ (3,1) (3,2) (3,3)\}$$

$R = \{(1,1) (2,2) (3,3)\}$ Here 'R' is Reflexive Relation

$R_1 = \{(1,1) (2,2)\}$ is not Reflexive because it does not contain (3,3)
($\forall x \in A$ xRx should be true)

$$R_2 = \{(1,1) (2,2) (3,3) (1,2)\} = \text{Reflexive Relation}$$

$\therefore$ Let $A = \{1, 2, \dots, n\} \rightarrow n$ elements

$R_n = \{(1,1) (2,2) \dots (n,n)\} \rightarrow$ Then the smallest Relation which is reflexive contains 'n' elements.

Note: If 'R' is Reflexive then any superset of 'R' is Reflexive

The largest Reflexive Relation on 'A' is " $A \times A$ "

The smallest Reflexive Relation on 'A' contains 'n' elements

If $|A| = n$ then largest Reflexive Relation = $n \times n = n^2$ elements.
 $|A| = n$ then smallest Reflexive Relation = n elements.

3. EXAMPLE 1 ON REFLEXIVE RELATIONS

If $A = \{1, 2, 3, \dots, n\}$ then the no. of reflexive relations possible on 'A'?

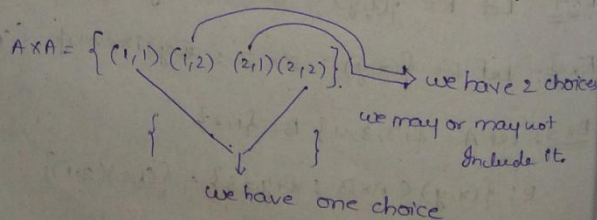
$$A = \{1, 2\}$$

$$R_1 = \{(1,1) (2,2)\} \checkmark$$

$$R_2 = \{(1,1) (2,2) (1,2)\} \checkmark$$

$$R_3 = \{(1,1) (2,2) (2,1)\} \checkmark$$

$$R_4 = \{(1,1) (2,2) (2,1) (1,2)\} = A \times A \checkmark$$



$$A = \{1, 2, 3\}$$

$$R_1 = \{(1,1) (2,2) (3,3)\}$$

$\therefore$ If $|A| = n$

$$|A| = n$$

The No. of Rel

Non-

4. EXAMPLE 2

Now whenever an element from element 2 times.

Ex:

1) The relation

2) The relation

Real numbers

3) The Relation collection of

4) The Relation

5) $R: \{(x,y) \in$

ive
TAL

$$A = \{1, 2, 3\}$$

$$R_1 = \{(1,1) (2,2) (3,3)\}$$

$$\begin{matrix} \underline{2} & \underline{2} & \underline{2} \\ \underline{2} & \underline{2} & \underline{2} \end{matrix}$$

1 way

$= 2^6$ combinations

$\therefore$ The no. of Reflexive relations
 $= 64$.

$\therefore$ If $|A|=n$ then the no. of Reflexive Relations on $A = 2^{n^2-n}$

$$|A|=n \Rightarrow \text{NO. of Reflexive Relations on } A = 2^{n(n-1)}$$

The NO of Relations that are not Reflexive = Total Relations - Reflexive Relations

$$= 2^{n \times n} - 2^{n^2-n}$$

$$\text{Non- Reflexive Relations from } A \text{ to } A = 2^{n^2} - 2^{n^2-n}$$

4. EXAMPLE 2 ON REFLEXIVE RELATIONS

Now whenever they ask if a Relation is Reflexive then try to take an element from the relation and apply the condition between the same element 2 times.

Ex:

1) The relation $\leq$ is reflexive on any set of Real numbers ($x \leq x$)

2) The relation 'is a divisor of' is reflexive on a set of non-zero Real numbers (x/x) $\forall x$ is divisible by $x = \text{TRUE} \Rightarrow$ Relation is Reflexive

3) The Relation is a subset of denoted by ' $\subseteq$ ' is reflexive on any collection of sets ($A \subseteq A$) $\Rightarrow \text{TRUE} \Rightarrow$ Reflexive

4) The Relation 'is parallel to' is reflexive on a set of all lines ($L \parallel L$) $= \text{TRUE}$

5) $R: \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : x-y \text{ is even integer}\}$ ($x-x=0 = \text{even} = \text{TRUE}$) $\Rightarrow$ Reflexive

have 2 choices
may not
include it.

5. Example 3 ON REFLEXIVE RELATIONS

Which of the following is False?

- a) If R_1 is Reflexive then every superset of R_1 is Reflexive = TRUE
 b) If R_1 is Reflexive, then subset of R_1 is reflexive = FALSE
 c) If R_1, R_2 are Reflexive then $R_1 \cap R_2$ is Reflexive = TRUE
 d) If R_1, R_2 are Reflexive then $R_1 \cup R_2$ is Reflexive = TRUE

(A)

$$R = \{(1,1), (2,3)\}$$

$$R_1 = \{(1,1), (2,2), (3,3)\}$$

$$R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (2,3), (3,1)\}$$

(D) $R_1 = \{(1,1), (2,2), (3,3)\}$

$$R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$$

$$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$$

= Reflexive

(B) $R = \{(1,2), (3)\}$

$$R_1 = \{(1,1), (2,2), (3,3)\}$$

$$R_2 = \{(1,1)\}$$

→ Not Reflexive

(C)

$$R_1 = \{(1,1), (2,2), (3,3)\}$$

$$R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$$

$$R_1 \cap R_2 = \{(1,1), (2,2), (3,3)\}$$

= Reflexive.

6. IRREFLEXIVE RELATIONS

Irreflexive Relations:

The Relation 'R' on a set 'A' is called irreflexive if 'x' is not related to 'x' i.e. $x \not R x \forall x \in A$ i.e. the ordered pair $(x,x) \notin R \forall x \in A$

Ex:

If a Relation is Not reflexive then we think that it is irreflexive but this is wrong assumption.

Ex: $A = \{1, 2, 3\}$

$$R_1 = \{(1,1), (2,2), (3,3)\}$$

Reflexive, NOT Irreflexive

$$R_2 = \{\}$$

NOT Reflexive, Irreflexive

$$R_3 = \{(1,1)\}$$

NOT Reflexive, Not Irreflexive

$$R_4 = \{(1,2), (2,1)\}$$

NOT Reflexive, Irreflexive

There does not exist a relation which is both Reflexive and Irreflexive

⇒ There will

⇒ The min-3 of a Irreflex

Now, $R_5 = A$
 $= \{$

Now $(A \times A) -$

If $|A| = n$ the

7. Example

Now, I want to

⇒ then $|A \times A|$

Let $A = \{1, 2, 3\}$

$$A \times A = \{(1,1),$$

$$(2,1),$$

$$(3,1),$$

∴ In general

∴ The total over a set

③ ⇒ There will be some relations which are neither reflexive nor Irreflexive. (9/25)

⇒ The min. set that is Irreflexive is $\{ \}$ and the min cardinality of a Irreflexive Relation is "Zero".

Now, $R_5 = A \times A$
 $= \{ (1,1) (1,2) (1,3) (2,1) (2,2) (2,3) (3,1) (3,2) (3,3) \}$

Now, Remove the diagonal elements because they make the Relation Reflexive

Now $\{ (A \times A) - (\text{diagonal elements}) \} = \{ (1,2) (1,3) (2,1) (2,3) (3,1) (3,2) \}$ This is the largest Irreflexive Relations

If $|A|=n$ then the cardinality of largest Irreflexive Relation will be $(n^2 - n)$

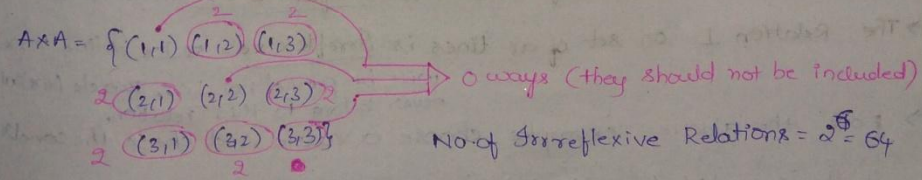
$\{ \text{Total no. of elements in } A \times A = (n \times n) = n^2 \} - \{ \text{Diagonal elements} \}$

7. EXAMPLE 1 ON IRREFLEXIVE RELATIONS

Now, I want to find no. of Irreflexive Relations on 'A' given $|A|=n$

⇒ then $|A \times A|$ contains n^2 elements

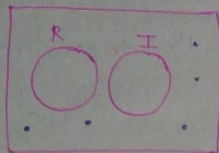
let $A = \{1, 2, 3\}$



$\therefore$ In general $A \times A = \{ \underbrace{(1,1) (2,2) (3,3) \dots (n,n)}_{n \text{ elements}}, \underbrace{(1,2) (1,3) \dots (n^2 - n)}_{n^2 - n \text{ elements}} \}$

∴ The total no. of Irreflexive relations that can be formed over a set of 'n' elements is $2^{n^2 - n}$

Now it is clear that



R = Reflexive Relations.
I = Irreflexive Relations

(10) (2)

Now, No. of Relations which are either reflexive or irreflexive is $2^{n^2-n} + 2^{n^2-n}$

$\therefore$ No. of Relations that are either Reflexive or Irreflexive 2^{n^2-n+1}

Now, No. of Relations that are neither Reflexive nor Irreflexive =
(Total) - (No. of Reflexive + Irreflexive)
 $= 2^{n^2} - (2^{n^2-n} + 2^{n^2-n})$

$\therefore$ No. of Relations that are Neither reflexive nor Irreflexive $= 2^{n^2} - 2^{n^2-n+1}$

8. EXAMPLE 2 ON IRREFLEXIVE RELATIONS

- The Relation ' $<$ ' on set of all Real numbers is Irreflexive = TRUE
proper subset
 - The Relation ' $\subset$ ' on set of all sets is "Irreflexive" = $A \subset A = \text{FALSE}$
 - The Relation ' $\perp$ ' on set of all lines is "Irreflexive" = $\alpha \perp \alpha$ (A line α , α will be parallel not $\perp$ i.e. two line (α , α) will never belong to this relation.)
- $\Rightarrow$ For the above examples choose a value and check the condition if it is valid then the Relation is Not Irreflexive if it fails then it is Irreflexive.

9. EXAMPLE 3 ON IRREFLEXIVE RELATION

- which of the following is false?
- Every subset of Irreflexive relation is irreflexive
- Every superset of Irreflexive relation is Irreflexive
- If R_1 is Irreflexive, R_2 is Irreflexive then $R_1 \cap R_2$ is irreflexive
- If R_1, R_2 are irreflexive then $R_1 \cup R_2$ is Irreflexive

Ans

A) $A = \{1, 2, 3\}$

$R = \{(1, 2), (2, 3), (3, 1)\}$

$R_1 = \text{Subset}$
 $= \{(1, 2), (1, 1)\}$

TRUE

10. EXAMPLE

State TRUE

a) The set of all

b) " " " "

c) " " " "

d) The set of

e) " " " "

f) " " " "

g) " " " "

h) " " " "

i) " " " "

j) " " " "

k) " " " "

l) " " " "

m) " " " "

n) " " " "

o) " " " "

p) " " " "

q) " " " "

r) " " " "

s) " " " "

t) " " " "

u) " " " "

v) " " " "

w) " " " "

x) " " " "

y) " " " "

z) " " " "

AA) " " " "

AB) " " " "

1) $A = \{1, 2, 3\}$

$$R = \{(1,2)(1,3) \\ (2,3)(2,1) \\ (3,1)(3,2)\}$$

$R_1 = \text{Subset of } R$

$$= \{(1,2)(1,3)\} = \text{IRREFLEXIVE}$$

TRUE

③

$$A = \{1, 2, 3\}.$$

$$R = \left\{ \begin{array}{l} (1,2) (1,3) (2,3) \\ (2,1) (3,1) (3,2) \\ (1,1) (2,2) \end{array} \right\}$$

IR-Relation

= NOT IRREFLEXIVE

10

$$A = \{1, 2, 3\}$$

$$R_1 = \{(1,2)(1,3)\} \quad R_2 = \{(2,1)(2,3)\}$$

$$\Rightarrow R_1 \cap R_2 = \{ \} = \phi$$

= IRREFLEXIVE

① TRUE (use same example)

10. EXAMPLE ON CLOSURE PROPERTIES

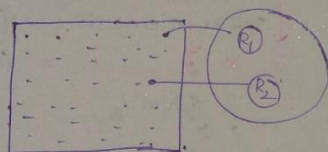
State TRUE / FALSE

9) The set of all reflexive relations is closed under the operation set union.

3) The set of all isomorphic relations are closed under the operation set union.

Intersection

Intersection



Set of all Reflexive Relations

II. SYMMETRIC RELATIONS

A Relation 'R' on a set 'A' is said to be symmetric if $(xRy) \Rightarrow (yRx) \forall$

* $x, y \in A$ i.e. if the ordered pair $(x, y) \in R$ then $(y, x) \in R \nmid x, y \in A$.

$$A = \{1, 2, 3\}$$

$R_1 = \{(1,2)(2,1)\} \rightarrow \text{Symmetric} \checkmark$

Reflexive $\times$

goreflexive ✓

$$R_3 = \{(2,3)\} \rightarrow \text{Not symmetric}$$

Not Reflexive

~~not~~ Soreflexive

$$R_4 = \{(3,1)(1,1)\} \text{ - Not symmetric}$$

- Not Reflexive

- Not Irreflexive

$$R_2 = \{(1,1)\} - \text{symmetric}$$

Not Reflexive

Not reflexive

$R = \{ \}$ - symmetric

The cardinality of the smallest symmetric Relation is Zero

$R = A \times A$ is symmetric $\therefore$ The largest cardinality is " n^2 " elements and

The largest symmetric relation that is defined on set A is " $A \times A$ "

12. NUMBER OF SYMMETRIC RELATIONS

$A = n$ then how many Relations are possible that are symmetric on $A \times A$

$|A| = n$ then $|A \times A| = n^2$

$A = \{1, 2, 3\}$

$$A \times A = \{(1,1)(2,2)(3,3) \quad (1,2)(2,1) \quad (1,3)(3,1) \quad (2,3)(3,2)\}$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad = 2^6$

$|A| = n = \{1, 2, 3, \dots, n\}$

$$(A \times A) = \{(1,1)(2,2)(3,3) \dots (n,n) \quad (1,2)(2,1) \quad (2,3)(3,2) \dots\}$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $2 \times 2 \times 2 \times \dots \times 2 \quad \times 2 \quad \times 2 \quad \times 2 \quad \times 2 \quad \times 2$
 $= 2^n$

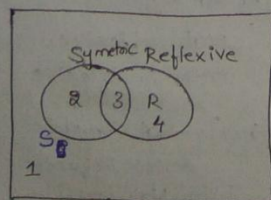
$\rightarrow$ NO. of pairs that will be formed.

$$(n^2 - n) / 2$$

$$\therefore \text{No. of symmetric Relations} = 2^n \times 2^{(n^2 - n) / 2} = 2^{n + \frac{n^2 - n}{2}}$$

13. EXAMPLE 1 ON SYMMETRIC RELATIONS

$|A| = n$, now I want to find the relation between the no. of Symmetric Relations and Reflexive Relations.



$2^{A \times A}$ } subset of powerset (total no. of relations).

Let $A = \{1, 2, 3\}$

$$R_1 = \{(1,2)(2,1)\}$$

$$R_2 = \{(1,1)(2,2)(3,3)\}$$

$$R_3 = \{(1,1)(2,2)(3,3)(1,2)\}$$

$$R_4 = \{(1,2)\}$$

15. Example

- State which
- Every set
 - " Sym
 - If R_1 a
 - " "
 - " "

(12) (18)

W.K.T the no. of symmetric Relations possible are $2^{\frac{n(n+1)}{2}}$

Now, set of all Relations that are both Symmetric and Reflexive = Region 3

is and

$$A \times A = \left\{ \overbrace{(1,1) (2,2) (3,3) \dots (n,n)}^{n^2 \text{ elements}}, \underbrace{(1,2) (2,1) \dots}_{n^2 - n \text{ elements}} \right\}$$

1 way because the Relation should be reflexive (n elements)

$$\therefore |S \cap R| = 2^{\frac{n(n+1)}{2}} = \text{Symmetric and Reflexive}$$

$$\text{Now, } |S - R| = |n(S)| - |n(S \cap R)|$$

$$= 2^{\frac{n(n+1)}{2}} - 2^{\frac{n(n+1)}{2}}$$

$$|S - R| = \text{Symmetric but not Reflexive}$$

$$= 2^{\frac{n(n+1)}{2}} - 2^{\frac{n(n+1)}{2}}$$

$$\text{Now, } |R - S| = |n(R)| - |n(S \cap R)|$$

$$|R - S| = 2^{\frac{n(n-1)}{2}} - 2^{\frac{n(n-1)}{2}}$$

Now the Relations that are not Symmetric and Reflexive = $n|S \cup R|$

$$= 2^{\frac{n^2}{2}} - \left(n(S) + n(R) - n(S \cap R) \right)$$

$$= 2^{\frac{n^2}{2}} - \left(2^{\frac{n(n+1)}{2}} + 2^{\frac{n(n-1)}{2}} - 2^{\frac{n(n-1)}{2}} \right)$$

$$\therefore \text{The relations that are not both Reflexive and Symmetric is } 2^{\frac{n^2}{2}} - \left(2^{\frac{n(n+1)}{2}} + 2^{\frac{n(n-1)}{2}} - 2^{\frac{n(n-1)}{2}} \right)$$

15. Example 3 ON SYMMETRIC RELATIONS

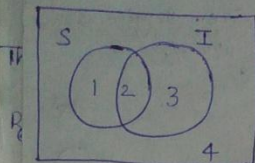
State which of the following are TRUE and False?

f Symmetric

- Every subset of symmetric Relation is symmetric (F)
- " superset " " " " (FALSE)
- If R_1 and R_2 are symmetric then $R_1 \cap R_2$ is symmetric (TRUE)
- " " " " " " " $R_1 \cup R_2$ " " (TRUE)
- " " " " " " " $R_1 - R_2$ " " (TRUE)

(1,2)

16. RELATION BETWEEN SYMMETRIC AND IRREFLEXIVE RELATION 17. ANTISY



- $R_1 = \{(1,1)\} \rightarrow$ Symmetric $\checkmark$ Not Irreflexive
- $R_2 = \{(1,2)(2,1)\} \rightarrow$ Symmetric $\checkmark$ Irreflexive $\checkmark$
- $R_3 = \{(1,2)\} \rightarrow$ Symmetric $\times$ Irreflexive $\checkmark$
- $R_4 = \{(1,2)(1,1)\} \rightarrow$ Symmetric $\times$ Irreflexive $\times$

WKT $n(S) = 2^{\frac{n(n+1)}{2}}$
 $n(I) = 2^{n^2-n}$

Now, $n(S \cap I) =$ Both Symmetric and Irreflexive

$$A \times A = \{(1,1)(2,2) \dots (n,n) \underbrace{(1,2)(2,1)}_{\substack{\downarrow 2 \\ \downarrow 2 \\ \downarrow 2 \dots \frac{(n^2-n)}{2} \text{ times}}}\}$$

$$\therefore n(S \cap I) = 2^{(n^2-n)/2}$$

$$n(S \cup I) = n(S) + n(I) - n(S \cap I) \rightarrow \text{Either Symmetric or Reflexive}$$

The no. of Relations that are Either Symmetric or Irreflexive $= \left(2^{\frac{n(n+1)}{2}} + 2^{n^2-n} \right) - 2^{(n^2-n)/2}$

$$n(S-I) = n(S) - n(S \cap I)$$

$\Rightarrow$ The no. of Relations that are Symmetric but not Irreflexive $= 2^{\frac{n(n+1)}{2}} - 2^{(n^2-n)/2}$

$$n(I-S) = n(I) - n(S \cap I)$$

The no. of Relations that are Irreflexive but not Symmetric is $= 2^{n^2-n} - 2^{(n^2-n)/2}$

$n(\overline{I \cup S}) =$ Relations that are neither reflexive nor symmetric $= n(U) - n(I \cup S) = 2^n - n(I \cup S)$

(14) A Relation $\forall x, y \in A$

$$A = \{1, 2, \dots\}$$

Now, $A \times A = \{(1,1), (1,2), (2,1), (2,2), \dots\}$

Now, if $A \times A =$

$\therefore$ The max ca

RELATION

7. ANTISYMMETRIC RELATION

(14) A Relation 'R' is said to be Antisymmetric if $(xRy \text{ and } yRx) \Rightarrow x=y$
 $\forall x, y \in A$.

$$A = \{1, 2, 3\}$$

$$R_1 = \{(1, 2), (2, 1)\} \rightarrow \text{NOT Antisymmetric because we have}$$

if $(x, y) \in R$ then $(y, x) \in R$.

$$R_2 = \{(1, 1)\} \rightarrow \text{Antisymmetric (This is the only exception)}$$

(the pairs of type (x, x) are allowed).

Symmetric $\checkmark$ (we cant say Symmetric & Antisymmetric are Compliment to each other).

$$R_3 = \{(1, 2), (1, 1)\} \Rightarrow \text{Not symmetric but it is Antisymmetric}$$

$$R_4 = \{(1, 2), (2, 1), (2, 3)\} \Rightarrow \text{Not Antisymmetric. } (x, y) \in R \text{ \& } (y, x) \in R.$$

Not Symmetric $(3, 2)$ is not present.

$$R_5 = \{(1, 1), (2, 2), (3, 3)\} \rightarrow \text{Both Symmetric and Antisymmetric (}$$

Exceptional case).

$$R_6 = \{\} - \text{Antisymmetric } \checkmark$$

The min cardinality of Antisymmetric Relation = 0

$$R_7 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 1)\} = \text{largest Antisymmetric Relation}$$

$$\text{Now, } A \times A = \{(1, 1), (2, 2), (3, 3)$$

$$(1, 2), (2, 1)$$

$$(2, 3), (3, 2)$$

$$(3, 1), (1, 3)\}$$

n^2 elements.

$$\text{Now, if } A \times A = \{(1, 1), (2, 2) \dots (n, n), (1, 2), (2, 1), (3, 1), (1, 3) \dots\}$$

n elements

$n^2 - n$ elements.

choose All

choose $\frac{n^2 - n}{2}$ ele

$$\therefore \text{The max cardinality of Antisymmetric Relation} = \left(n + \frac{n^2 - n}{2}\right)$$

18. NUMBER OF ANTISYMMETRIC RELATIONS

$$|A|=n \quad |A \times A|=n^2$$

$$A = \{1, 2, 3\} \quad A \times A = \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{matrix} \right\}$$

= $2^3 \times 3^3$ Relations
= 2 diagonals $\times$ 3 nondiagonal pairs

- case 1: Include (1,2)
- 2: Include (2,1)
- 3: Include None

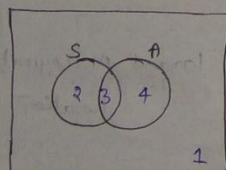
∴ No. of Antisymmetric Relations possible on A where $|A|=n$ is

$$= \binom{n}{2} \times 2^{\frac{n^2-n}{2}}$$

$$A \times A = \left\{ \underbrace{(1,1) \dots (n,n)}_n, \underbrace{(1,2) \dots (2,1)}_3, \underbrace{(3,2) \dots (2,3)}_{3 \dots \frac{n^2-n}{2} \text{ times}} \right\}$$

⇒ $2^n \times 2^{\frac{n^2-n}{2}}$ Relations

19. RELATION BETWEEN SYMMETRIC AND ANTISYMMETRIC RELATIONS



$$|U| = 2^n$$

$$n(S) = 2^{n(n+1)/2}$$

$$n(A) = 2^n \times 2^{\frac{n^2-n}{2}}$$

$$R_1 = \{(1,2) (2,1) (2,3)\} \rightarrow \text{Antisymmetric} \times \text{Symmetric} \times$$

$$R_2 = \{(1,2) (2,1)\} \rightarrow S \checkmark \quad AS \times$$

$$R_3 = \{\} \rightarrow AS \checkmark \quad S \checkmark$$

$$R_4 = \{(1,1)\} \rightarrow AS \checkmark \quad S \checkmark$$

$$R_5 = \{(2,1)\} \rightarrow AS \checkmark \quad S \times$$

$$A \times A = \left\{ \underbrace{(1,1) (2,2) \dots (n,n)}_{\text{Choose All}}, (1,2) (2,1) \dots \right\}$$

$$= 2^n$$

The no. of Relations that will be Both symmetric and Antisymmetric = 2^n → only diagonal elements

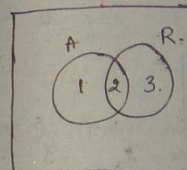
Now, the no. of relations

$$n(S \cup A)$$

$$n(S \cup A)$$

Similarly $n(S)$

20. RELATION



$$n(A) = 2^n \times 2^{n(n-1)/2}$$

$$n(R) = 2^n \times 2^{n(n-1)/2}$$

$$n(A \cap R) = 2^{n^2}$$

$$A \times A = \{(1,1) (1,2) (2,1) (2,2)\}$$

Both Reflexive &

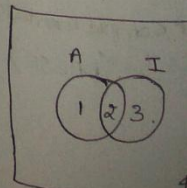
$$n(A \cup R) = n(A) + n(R) - n(A \cap R)$$

$$n(A - R) = n(A) - n(A \cap R)$$

$$n(R - A) = n(R) - n(A \cap R)$$

$$n(\overline{A \cap R}) = n(U) - n(A \cap R)$$

RELATION B



$$n(U) = 2^{n^2}$$

(16)

Now, the no. of relations that are either Symmetric / Antisymmetric are

$$n(SUA) = n(S) + n(A) - n(S \cap A)$$

$$n(SUA) = \binom{n(n+1)}{2} + \binom{n(n-1)}{2} - \binom{n}{2}$$

Relations

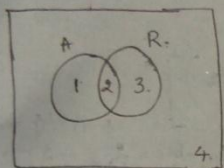
Nondisjoint pairs

Similarly $n(S-A)$ and $n(A-S)$ can be found

$$n(S-A) = n(S) - n(S \cap A)$$

$$n(A-S) = n(A) - n(S \cap A)$$

20. RELATION BETWEEN REFLEXIVE AND ANTISYMMETRIC RELATIONS



$$n(A) = 2^n \cdot 3^{n(n-1)/2}$$

$$n(R) = 2^{n(n-1)/2}$$

$$n(U) = 2^{n^2}$$

$$R_1 = \{(1,2)\}$$

$$R_2 = \{(1,1), (2,2), \dots, (n,n)\}$$

$$R_3 = \{(1,1), (2,2), \dots, (n,n), (1,2), (2,1)\}$$

$$R_4 = \{(2,1), (1,2)\}$$

$$A \times A = \{(1,1), (2,2), \dots, (n,n), (1,2), (2,1)\}$$

$$\text{Both Reflexive \& Antisymmetric} = 1 \times 3$$

$$= \frac{n(n-1)/2}{3} = n(ANR)$$

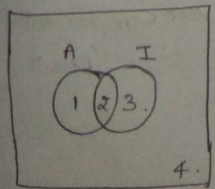
$$n(AUR) = n(A) + n(R) - n(ANR) = \text{Both Antisymmetric and Reflexive}$$

$$n(A-R) = n(A) - n(ANR) = \text{Antisymmetric but not Reflexive}$$

$$n(R-A) = n(R) - n(ANR) = \text{Reflexive but not Antisymmetric}$$

$$n(\overline{A \cup R}) = n(U) - n(R \cup A) = \text{Not Reflexive and not Antisymmetric}$$

RELATION BETWEEN IRREFLEXIVE AND ANTISYMMETRIC RELATIONS



$$n(U) = 2^{n^2}$$

$$n(A) = 2^n \cdot 3^{n(n-1)/2}$$

$$n(I) = 2^{n(n-1)/2}$$

$$R_1 = \{(1,1)\} \Rightarrow A \checkmark \quad IR \times$$

$$R_2 = \{(1,2)\} \Rightarrow A \checkmark \quad IR \checkmark$$

$$R_3 = \{(1,2), (2,1)\} \Rightarrow A \times \quad IR \checkmark$$

$$R_4 = \{(1,1), (1,2), (2,1)\} \Rightarrow A \times \quad IR \times$$

$$A \times A = \{(1,1) (2,2) \dots (n,n) (1,2) (2,1) (2,3) (3,2) \dots\}$$

(8)

The Relations that are Both Reflexive and Antisymmetric

$$A \times A = \{(1,1) (2,2) \dots (n,n) (1,2) (2,1) (2,3) (3,2) \dots\}$$

If I include this
they become reflexive
so don't include them

3 choices 3 choices
 $(n^2 - n)/2$
3

The no. of Relations that are Both Symmetric and Reflexive is $\frac{(n^2 - n)/2}{2} = 3$

$$\text{Now, } n(A-I) = n(A) - n(A \cap I) = \left[\begin{pmatrix} n & n(n-1)/2 \\ 2 & 3 \end{pmatrix} \right]$$

$$n(I-A) = n(I) - n(A \cap I) = \text{Irreflexive but not Antisymmetric} = \frac{n(n-1)}{2} - 3$$

$$n(A \cup I) = n(A) + n(I) - n(A \cap I) = \text{No. of Antisymmetric but not Reflexive} = \left(\frac{n(n-1)}{2} - 3 \right)$$

$$n(\overline{A \cup I}) = n(U) - n(A \cup I) = \frac{n^2}{2} - \left(\frac{n(n-1)}{2} - 3 \right)$$

22. ANTISYMMETRIC PROPERTIES

State TRUE / FALSE

a) Every subset of Antisymmetric relation is Antisymmetric (TRUE)

b) " Superset " " " " " " (FALSE)

c) Antisymmetric relations are closed under set union. (FALSE)

d) " " " " " " " Intersection (TRUE)

e) " " " " " " " difference (TRUE)

f) " " " " " " " Set Complementation (FALSE)

23. EXAMPLE

1) The Relation

2) The Relation

3) The Relation

on any set

4) The Relation

17. $x \leq y$ then

is Antisymmetric

27. $x < y$ then

therefore the

37. x/y (i me

$\hookrightarrow x$ is di

2/4 (4

47. Antisymmetric

24. ASYMMETRIC

A Relation R

$(y \not R x) \forall x, y$

$A = \{1$

$R_1 = \{$

$R_2 =$

Note:

$R = S$

(18)

23. EXAMPLES ON ANTI SYMMETRIC RELATION

- 1) The Relation ' $\leq$ ' is Antisymmetric on any set of Real numbers.
- 2) The Relation ' $<$ ' is Antisymmetric on any set of Real numbers.
- 3) The Relation "is a divisor" of denoted as ' $/$ ' is an antisymmetric on any set of the Real numbers.
- 4) The Relation ' $\subseteq$ ' (set inclusion) is Antisymmetric on any collection of sets.
 - 1) $x \leq y$ then (y/x) will not be present and $(y \leq x)$ is false $\therefore$ The relation is Antisymmetric.
 - 2) $x < y$ then $(y < x)$ is false, so (y/x) will not be present in Relation $\therefore$ therefore the relation is Antisymmetric.
 - 3) x/y (i mean if y is divisible by x).
 - $\hookrightarrow x$ is divisor of y then y won't be divisor of x .
 - $2/4$ (4 is $\div$ ble by 2) then $(4/2)$ (2 won't be $\div$ ble by 5).
- 4) Antisymmetric.

24. ASYMMETRIC RELATIONS

A Relation ' R ' on set ' A ' is said to be Asymmetric if (xRy) then $(y \not R x) \forall x, y \in A$

$$A = \{1, 2, 3\}$$

$R_1 = \{(1, 2)\}$ Asymmetric Relation (because (x, y) is present but (y, x) is not present).

$R_2 = \{(1, 2), (2, 2)\}$ Not Asymmetric (because we included x, x)

Antisymmetric ($\checkmark$) because (x, y) is present and (y, x) is not present and (i, i) is allowed case

Note: Diagonal elements they can be present in Antisymmetric but not in Asymmetric

$R_3 = \{ \}$ Asymmetric, Antisymmetric, symmetric

The min. cardinality of smallest Asymmetric Relation = '0'.

$R_4 = \{(1,2) (2,1)\} \rightarrow$ Antisymmetric $\times$ Asymmetric $\times$ Symmetric $\checkmark$

(20)

$A \times A = \{(1,1) (2,2) (3,3) (1,3) (3,1) (1,2) (2,1) (2,3) (3,2)\}$ $R_5 = \{(1,3) (1,2) (2,3)\}$ - This is the largest asymmetric Relation.

Now, $A \times A = \{(1,1) (2,2) \dots (n,n) (1,2) (2,1) (3,2) (2,3) \dots\}$
 $\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $\times \quad \times \quad \times \quad \times$
 They violate the Asymmetric property. Include one of $(1,2) (2,1)$

$\therefore$ The cardinality of the largest Asymmetric Relation that is possible with a set with 'n' elements is $= \left(\frac{n^2-n}{2} \right)$

25. NO OF ASYMMETRIC RELATIONS

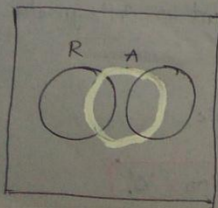
$|A|=n \quad |A \times A| = n \times n$

Now, $|A \times A| = \{(1,1) (2,2) \dots (n,n) (1,2) (2,1) (3,2) (2,3) (1,3) (3,1) \dots\}$
 $\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 should not include 3×1 3×1 3×1 it
 $= 3 \times 3 \times 3 \dots \left(\frac{n^2-n}{2} \right) \text{ times.}$

$\therefore$ The no. of Asymmetric Relations $= 3^{\left(\frac{n^2-n}{2} \right)}$

26. REFLEXIVE AND ASYMMETRIC RELATIONS

$\rightarrow$ If a Relation is Reflexive then it cannot be Asymmetric



$n(A) = 3^{\frac{n(n-1)}{2}}$

$n(R) = 2^{\frac{n(n-1)}{2}}$

$n(A \cup R) = n(A) \cup n(R) = n(A) + n(R)$

$n(A \cap R) = 0$

$n(A - R) = n(A)$

$n(R - A) = n(R)$

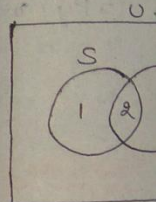
$n(\overline{R \cup A}) = n(U) - n(R \cup A)$

27. REFLEXIVE

$\rightarrow$ Every Asym

$\rightarrow$ Every Asym

28. SYMMETRIC



Now, $n(S) \in A$

$n(S \cup A)$

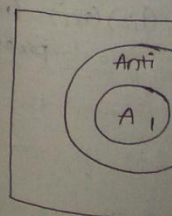
$n(S \cap A)$

$n(S - A)$

$n(A - S)$

29. ANTISYMMETRIC

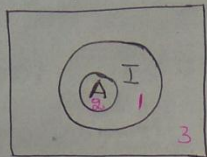
Every Asymmet



27. IRREFLEXIVE AND ASYMETRIC RELATIONS

⇒ Every Asymmetric Relation is Irreflexive but

⇒ Every Irreflexive Relation is not Asymmetric



$$n(I) = 1$$

$$n(A) = 2$$

$$n(I \cup A) = n(I) + n(A) = 3$$

$$n(I \cap A) = 0$$

$$n(I - A) = n(I) = 1$$

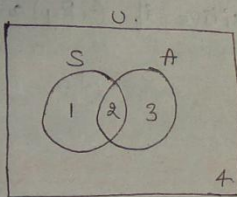
$$n(A - I) = n(A) = 2$$

$$n(\overline{I \cup A}) =$$

$$= n(U) - n(I \cup A) = 3 - 3 = 0$$

$$= n(U) - n(I) - n(A) = 3 - 1 - 2 = 0$$

28. SYMMETRIC AND ASYMMETRIC RELATIONS



$$R_1 = \{(1,1)\} \quad S \checkmark \quad AS \times$$

$$R_2 = \{(1,2)\} \quad S \checkmark \quad AS \checkmark$$

$$R_3 = \{(1,2), (2,1)\} \quad S \times \quad AS \checkmark$$

$$R_4 = \{(1,1), (1,2)\} \quad S \times \quad AS \times$$

$$n(S) = 2$$

$$n(A) = 2$$

$$n(S \cap A) = 1$$

$$\text{Now, } n(S) \in n(S) \rightarrow n(S \cap S) = n(S) = 2$$

$$n(S \cup A) = n(S) + n(A) - n(S \cap A) = 2 + 2 - 1 = 3$$

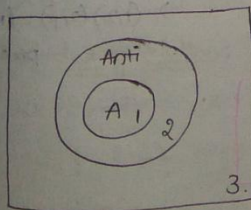
$$n(\overline{S \cup A}) = n(U) - n(S \cup A) = 4 - 3 = 1$$

$$n(S - A) = \{n(S) - n(S \cap A)\} = 2 - 1 = 1$$

$$n(A - S) = \{n(A) - n(S \cap A)\} = 2 - 1 = 1$$

29. ANTISYMMETRIC AND ASYMMETRIC RELATIONS

Every Asymmetric Relation is Antisymmetric



$$R_1 = \{(1,1)\} \quad \text{Antisymmetric} \checkmark \quad \text{Asymmetric} \times$$

$$R_2 = \{(1,2), (2,1)\} \quad \text{Antisymmetric} \times \quad \text{Asymmetric} \times$$

$$R_3 = \{(1,2)\} \quad \text{Antisymmetric} \checkmark \quad \text{Asymmetric} \checkmark$$

$$\text{Now, } n(A \cup \text{Anti}) = n(\text{Anti}) = 2$$

$$= n(\overline{A \cup \text{Anti}}) = n(U) - n(A \cup \text{Anti}) = 3 - 2 = 1$$

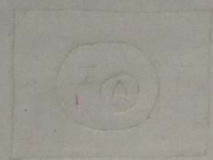
$$= n(\text{Anti} - \text{Asy}) = n(\text{Anti}) - n(\text{Asy}) = 2 - 1 = 1$$

$$= n(\text{Asy} - \text{Anti}) = 0$$

30. PROPERTIES OF ASYMETRIC RELATIONS

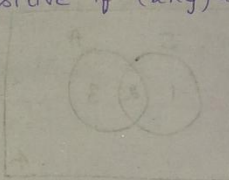
Asymmetric Relations are closed under

- a) subset operation (TRUE)
- b) Superset operation (FALSE)
- c) union " (FALSE)
- d) Intersection " (TRUE)
- e) Set difference (TRUE)
- f) Complementation (FALSE)



31. TRANSITIVE RELATIONS

A Relation R on set A is said to be transitive if (xRy) and (yRz) then $(xRz) \forall x, y, z \in A$



$$R_1 = \{ \} - \text{Transitive}$$

$$R_2 = \{ (1,1) \} - \text{Transitive}$$

$$R_3 = \{ (a,b), (c,d) \} \rightarrow \text{Transitive (we don't have } (x,y)(y,z) \text{ pair)}$$

$$R_4 = \{ (x,y), (y,z) \} \rightarrow \text{NOT Transitive } [(x,z) \text{ is absent}]$$

$$R_5 = \{ (x,y), (y,z), (x,z) \} = \text{Transitive}$$

$$R_6 = \{ (1,2), (2,2) \} - \text{Transitive}$$

$$R_7 = \{ (1,2), (2,1) \} - \text{Transitive} \Rightarrow (2,1)(1,1) \text{ is present}$$

$$R_8 = A \times A = \text{Transitive}$$

$$\Rightarrow (1,2)(2,1) = (1,1) \text{ present}$$

The largest relation which is Transitive = $A \times A$
 The smallest relation which is Transitive = $\{ \}$

32. EQUIVALENCE

A Relation 'R' on set 'A' is said to be an equivalence relation if 'R' is reflexive, symmetric and transitive.

$$\text{Ex: } A = \{a, b, c\}$$

$$R_1 = \{ (a,a), (b,b), (c,c) \}$$

$$R_2 = \{ (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b) \}$$

$$R_3 = \{ (a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b), (a,b,c), (b,c,a), (c,a,b) \}$$

The smallest

$\Rightarrow$ The largest and it is

33. EXAMPLES

Which of the following are equivalence relations on the set of all Real numbers?

$$a) R_1 = \{ (a,b) / a-b \text{ is an integer} \}$$

$$b) R_2 = \{ (a,b) / a+b \text{ is an integer} \}$$

$$c) R_3 = \{ (a,b) / a \leq b \}$$

$$d) R_4 = \{ (a,b) / a \text{ divides } b \}$$

34. POSET

Partial Ordering

A relation 'R' on a set 'A' is said to be a partial ordering if 'R' is reflexive, antisymmetric and transitive.

Partially ordered set

A set 'A' with a partial ordering 'R' is called a partially ordered set (poset).

32. EQUIVALENCE RELATION

- A Relation 'R' on a set 'A' is said to be Equivalence Relation on 'A' if 'R' is 1) Reflexive 2) Symmetric 3) Transitive

Ex: $A = \{a, b, c\}$

$$R_1 = \{(a, a), (b, b), (c, c)\}$$

$$R_3 = \{(a, a), (b, b), (c, c), (b, c), (c, b)\}$$

$$R_2 = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$$

$$R_4 = \{(a, a), (b, b), (c, c), (a, c), (c, a)\}$$

$$R_5 = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c), (c, a), (c, b), (b, a)\}$$

- The smallest Equivalence set/Relation on set A contains 'n' elements.
 → contains only diagonal elements.

- ⇒ The largest Equivalence Relation on set A contains 'n²' elements and it is 'A x A'.

33. EXAMPLES OF EQUIVALENCE RELATIONS.

Which of the following is not an Equivalence relation on a set of all Real numbers?

a) $R_1 = \{(a, b) / a - b \text{ is an integer}\}$
 $\begin{matrix} R \checkmark \\ S \checkmark \\ T \checkmark \end{matrix}$

b) $R_2 = \{(a, b) / a - b \text{ is divisible by 5}\}$
 $\begin{matrix} R \checkmark \\ S \checkmark \\ T \checkmark \end{matrix}$

c) $R_3 = \{(a, b) / a - b \text{ is odd no.}\}$ → Not Reflexive (diagonal ele diff = 0 = Even no.)

d) $R_4 = \{(a, b) / a - b \text{ is an even no.}\}$
 $\begin{matrix} R \checkmark \\ S \checkmark \\ T \checkmark \end{matrix}$

$\left. \begin{matrix} S \text{ (Symmetric)} \\ T \text{ (Transitive)} \end{matrix} \right\} = \text{Equivalence Relation.}$

34. POSET

Partial Ordering Relation:

A relation 'R' on set A is said to be partial ordering relation (partial order) if 'R' is Reflexive, Antisymmetric, and Transitive

Partially ordered Set

A set 'A' with a partial order 'R' defined on 'A' is called partially Ordered set (poset) and it is denoted by $[A; R]$

$$A = \{1, 2, 3\}$$

$$R_1 = \{(1,1), (2,2), (3,3)\}$$

Reflexive ✓

Transitive ✓

Symmetric ✓

Antisymmetric ✓

The Relation is Equivalence and partial order Relation.

⇒ This is the smallest Relation which is both partial ordering and Equivalence Relation.

$R_2 = \{(1,1), (2,2)\} \rightarrow$ cannot be Equivalent and partial ordering (Because this doesnot contain $(3,3)$)

$$R_3 = \{(1,1), (2,2), (3,3), (1,2)\} \Rightarrow \text{Not Equivalent Relation}$$

Reflexive ✓

Antisymmetric ✓

Transitive ✓

partial ordering Relation.

Now, let us consider set of all Real numbers R . Now, let the Relation be " $\leq$ " then R is Reflexive, Antisymmetric, Transitive. Therefore

$[R; \leq]$ is called the partial order set / poset.

This Relation ($\leq$) is the partial order set on Relation ' R '.

Now, $[S; \subseteq]$ is also a poset — Necessary condition is the
 $[R; \subseteq]$ is also a poset — Relation should be Reflexive, Antisymmetric, Transitive.

35. TOS

Totally ordered set (Linearly ordered set or chain)

A poset $[A; R]$ is called a "Totally ordered set" if every pair of elements in ' A ' are comparable, i.e. aRb or $bRa \forall a, b \in A$

Ex: find whether the following are totally ordered sets or not

1) If ' A ' is any set of real nos then poset $[A; \leq]$ is TOS.

2) If $A = \{1, 2, 3, 4, \dots, 10\}$ then the poset $[A; \leq]$ is a TOS.

3) If $A = \{1, 2, 6, 30, 60, 300\}$ then $[A; \mid]$ is TOS [$2 \div 6$, $6 \div 300$, $60 \div 300$].

4) If $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ then $[S; \subseteq]$ is Not TOS.

5) If $S = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$ then $[S; \subseteq]$ is TOS.

EQUIVALENCE RELATION

(24)

Now, if i take a them, $\begin{cases} \text{if } i \text{ take } a \\ \text{if } i \text{ take } b \end{cases}$

$$47. \{a\} \not\subseteq \{b\}$$

$$57. \{a\} \subseteq \{a, b\}$$

$$\{a, b\} \subseteq \{a, b, c\}$$

36. GATE QUEST

Let $A = \{a, b, c\}$ wh

$$a) R_1 = \{(a, a), (c, c)\}$$

$$b) R_2 = \{(a, b), (b, a), (a, c)\}$$

$$c) R_3 = \{(a, b), (b, a), (c, c)\}$$

$$d) R_4 = \{(a, b), (b, c), (c, a)\}$$

37. GATE QUEST

Let $A = \{a, b, c, d\}$ c

$$(b, a), (b, b), (b, c), (b, d)$$

a) R is Equivalence

b) R is irreflexive

c) R is symmetric

d) R is transitive

38. GATE QUESTION 3

Let $A =$ set of all

$$\text{be } aRb \Leftrightarrow b = a^k$$

a) R is Equivalence

b) ' R ' is partial order

c) ' R ' is reflexive and

d) ' R ' is total order.

$$R = \{(2, 2), (2, 4), (2, 8), (2, 16), (4, 4), (4, 16), (8, 8), (8, 64), (16, 16), (16, 256)\}$$

24. Now, if i take any two Real nos, i can put the relation ' $\leq$ ' in b/w them, $\left\{ \begin{array}{l} \text{if i take } 1, 2 \text{ i can say } (1 \leq 2) \\ \text{if i take } 4, 2 \text{ i can say } (2 \leq 4) \end{array} \right\} \therefore \text{Every pair of Real nos is comparable}$
 $\Rightarrow$ This is "TOS".

47. $\{a\} \not\subseteq \{b\} \therefore$ This is not a "Total ordered set."

50. $\{a\} \subseteq \{a, b\}, \emptyset \subseteq \{a\}, \{a, b\}, \{a, b, c\}$
 $\{a, b\} \subseteq \{a, b, c\}$

36. GATE QUESTION-1

- Let $A = \{a, b, c\}$ which of the following is Not TRUE (FALSE)?
- a) $R_1 = \{(a, a), (c, c)\}$ is symmetric, Antisymmetric, Transitive (TRUE)
 - b) $R_2 = \{(a, b), (b, a), (a, c)\}$ is symmetric, Antisymmetric (FALSE) ((c, a) is not present)
 - c) $R_3 = \{(a, b), (b, a), (c, c)\}$ is symmetric but not Antisymmetric (TRUE) And
 - d) $R_4 = \{(a, b), (b, c), (c, c)\}$ is Antisymmetric but not symmetric. (TRUE)

37. GATE QUESTION 2

- Let $A = \{a, b, c, d\}$ and a relation on set A is defined as $R = \{(a, a), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d)\}$ which of the following is TRUE?
- a) R is Equivalence Relation = NOT EQUIVALENCE (NOT REFLEXIVE ((d, d))).
 - b) R is irreflexive or Antisymmetric relation FALSE (NOT IRREFLEXIVE)
 - c) R is symmetric or Asymmetric Relation (FALSE) ($\{a, b\}$ is absent)
 - d) R is transitive. (TRUE) ($(b, a), (a, a) \Rightarrow (b, a)$ and present in R)

38. GATE QUESTION 3

- Let $A =$ set of all Real numbers, $R = \{(a, b) / b = a^k \text{ for some integer } k\}$
 $a R b \Leftrightarrow b = a^k$
- 1) R is Equivalence Relation $\left\{ \begin{array}{l} R \checkmark \\ S \times \\ T \checkmark \end{array} \right. \therefore \text{Not Equivalent}$
- 2) R is partial order $\left\{ \begin{array}{l} R \checkmark \\ S \checkmark \\ T \checkmark \end{array} \right. \therefore \text{The Given Relation is partial order.}$
- 3) R is reflexive and symmetric but not Transitive
- 4) R is total order.
- $R = \{(2, 2), (2, 4), (2, 8), (2, 16), \dots\}$
 $\{(1, 1), (3, 3), (3, 9), (3, 27), \dots\}$
- NOT Symmetric = $(8, 2)$ is not present
 $\Rightarrow 2 = 8^{1/3} \rightarrow$ but k is Integer

39. GATE QUESTION 4

26

Which of the following is NOT TRUE (FALSE)?

- a) If a Relation 'R' on set A is symmetric and Transitive then 'R' is Reflexive
- b) If a relation 'R' on set A is irreflexive and transitive then 'R' is Antisymmetric
- c) If 'R' and 'S' are Antisymmetric on A then $(R \cup S), (R \cap S)$ are also Antisymmetric
- d) If R, S are Transitive then

a) $A = \{1, 2, 3\}$ $A = \{ \}$ Not Reflexive, but not (Symmetric & Transitive).
 $R = \{(1, 1)\}$ - Transitive, symmetric, not Reflexive $\Rightarrow$ op1: FALSE.

b) R = TRUE (c) FALSE (d) TRUE

41. GATE QUESTION 5

The no. of Equivalence relations on set $\{1, 2, 3, 4\}$ is

a) 15 b) 16 c) 24 d) 4: Just Remember if a set has

$$R = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

3 elements $\rightarrow$ No. of Equivalence Relations

= 5

4 elements $\rightarrow$ No. of Equivalence Relations = 15

SUMM

17 Refl

27 Sym

37 Sym

47 Anti

57 Asym

67 Trans

77 Equival

87 POSET

97 TOS

SUMMARY ON RELATIONS

1) Reflexive Relation: All the diagonal ele should definitely be present

2) Irreflexive Relation: No diagonal ele should be present if you find atleast one diagonal ele then it is NOT IRREFLEXIVE

3) Symmetric Relation: If (x, y) is present then only check for (y, x)
(All the symmetric pairs need not be present).

$$\{ \} - \text{symmetric} \begin{cases} \text{Min cardinality} = 0 \\ \text{Max cardinality} = n^2 \end{cases}$$

4) Anti symmetric Relation: Symmetric pairs should not be present but diagonal pairs are allowed (Exception case)

$$\{ \} - \text{Anti symmetric} \begin{cases} \text{Min cardinality} = 0 \\ \text{Max cardinality} = n + \frac{n^2 - n}{2} \end{cases}$$

5) Asymmetric Relation: $\rightarrow$ symmetric pairs should not be present, No exception on diagonal elements also.

$$\{ \} - \text{Asymmetric} \begin{cases} \text{Min cardinality} = 0 \\ \text{Max cardinality} = \frac{n^2 - n}{2} \end{cases}$$

6) Transitive Relation: If (a, b) is present and (b, c) is present then only check for (a, c) if (a, c) is present then Transitive else Not Transitive.

$$\Rightarrow \text{If } (a, b) \text{ and } (b, c) \rightarrow (a, c) \quad \left| \begin{array}{l} R = \{a, b, c\} \\ R_1 = \{(a, b), (b, c)\} \end{array} \right. \begin{array}{l} \text{not} \\ \text{Transitive} \end{array}$$

$$\Rightarrow \{ \} = \text{Transitive} \begin{cases} \text{Min cardinality} = 0 \\ \text{Max cardinality} = n^2 \text{ (AX4)} \end{cases}$$

7) Equivalence Relation: $\rightarrow$ Reflexive
 $\rightarrow$ Symmetric
 $\rightarrow$ Transitive } (TRS)

8) POSET: Reflexive, Anti symmetric, Transitive (RAT).

9) TOS: Every pair should be comparable on the Relation given.

3. PARTIAL ORDERS AND LATTICES

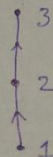
1. POSET DIAGRAM (HASSE DIAGRAM)

Let $[A; R]$ be a poset. The poset diagram is as follows

- 1) There is a vertex corresponding to each element of 'A'.
- 2) An edge between the elements 'a' and 'b' is not present in the diagram if there exists an element $x \in A$ such that (aRx) and (xRb) .
- 3) An edge b/w the elements 'a' and 'b' is present iff aRb and there is no element $x \in A$ such that (aRx) and (xRb) .

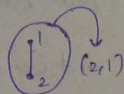
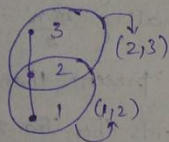
Ex: $A = \{1, 2, 3\}$

$$R = \leq = \{ (1,2), (1,3), (2,3), (1,1), (2,2), (3,3) \}$$



(1,3) is not needed
because $(1,2)(2,3) \Rightarrow (1,3)$, anyway (1,2) (2,3) edges are present

$\Rightarrow$ The general convention that we assume is bottom to top and we don't use arrows also.

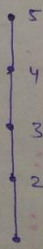


2. EXAMPLES ON POSET DIAGRAMS

$$A = \{1, 2, 3, 4, 5\} \quad \text{poset} = [A; \leq]$$

$$R = \leq$$

$$R = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5) \}$$



$\Rightarrow$ Here the diagram looks like chain
 $\Rightarrow$ The relation is TOR (Total Order Relation)

3. LUB

Least upper

Let $[A; R]$

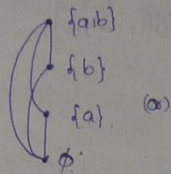
such that

ii)

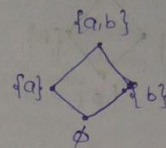
(28)

$$S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

$$\text{poset: } [S, \subseteq]$$

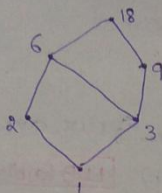


(29)



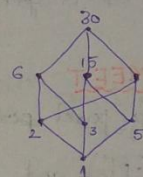
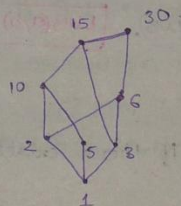
$$A = \{1, 2, 3, 9, 6, 18\}$$

$$\text{poset: } [A, |]$$



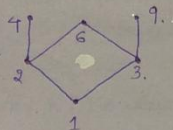
$$A = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

$$\text{poset: } [A, |]$$



$$A = \{1, 2, 3, 4, 6, 9\}$$

$$\text{poset: } [A, |]$$



3. LUB

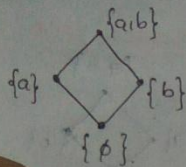
Least upper Bound (LUB or Join or Supremum)

Let $[A, R]$ be a poset. for $a, b \in A$, if there exists an element $c \in A$ such that $a R c$ and $b R c$,

ii) if there exists any other element d such that $(a R d)$ and $(b R d)$ then $(c R d)$, then c is the LUB of a and b .

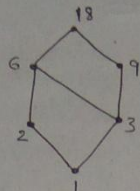


Ex:



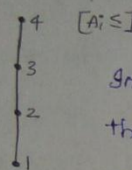
$$LUB = \{a, b\}$$

$[D_6, /]$



$$LUB = \{18\}$$

⇒ In case of divisor operation $LUB(a, b) = LCM(a, b)$.



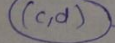
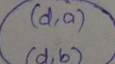
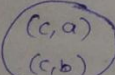
In case of $\leq$ Relationship if we are comparing (a, b) then the LUB will be $\max(a, b)$. $LUB(a, b) = \max(a, b)$
⇒ In this diagram $LUB = \{4\}$.

⇒ In case of set Inclusion Relation (subset or equal to Relation) then union of two sets is going to be (LUB). $LUB(a, b) = \{a \cup b\}$

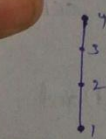
4. GLB OR MEET

Let $[A, R]$ be a poset, for $a, b \in A$ if there exists an element c such that

- (c, a) and (c, b) and
- If there exists any other element d such that (d, a) and (d, b) then (d, c) then c is called the GLB of a and b .

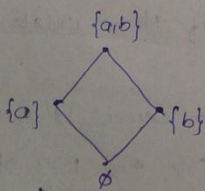


→ c is called the Greatest lower Bound.



$$GLB(a, b) = \min(a, b)$$

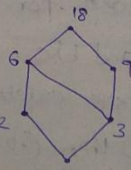
$\leq$ Relationship



$$GLB(a, b) = a \cap b$$

Set Inclusion

Subset / Equals



$$GLB(a, b) = GCD(a, b)$$

5. STANDARD

a) $[A; \leq]$ LUB = GLB =

b) $[A; /]$ LUB = GLB =

c) $[S; \subseteq]$ LUB = GLB =

6. LATTICE

Join semi lattice:

Meet semi lattice

Lattice: If Both

Ex: $A = \{1, 2, 3, \dots, 10\}$

$S = \{ \{a\}, \{b\}, \dots \}$

$A = \{1, 2, 3, 4, \dots\}$

do some thing like

... A TOS (Total Order Set)

7. LATTICE EXAMPLE

→ If A is set of all

→ If n is a positive

Ex: $D_6 = \{1, 2, 3, 6\}$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Every $[D_n, /]$ is a

→ If $p(A)$ denotes the

30

STANDARD EXAMPLES.

1) $[A; \leq]$ LUB $(a, b) = \max(a, b)$
GLB $(a, b) = \min(a, b)$

$A =$ set of Real numbers

2) $[A; /]$ LUB $(a, b) = \text{Lcm}(a, b)$
GLB $(a, b) = \text{GCD}(a, b)$

$A =$ set of Real no's.

3) $[S; \subseteq]$ LUB = Union
GLB = Intersection.

$S =$ set of all sets.

b) then

for a given pair of elements the LUB, GLB may or maynot exist.

6. LATTICE

Join semi lattice: If LUB exists for every pair of elements in poset.

Meet semi lattice: If GLB exists for every pair of elements in poset.

Lattice: If Both LUB and GLB " " " " " " " " " " " "

GLB exists for every pair.

Ex: $A = \{1, 2, 3, \dots, 10\}$ $[A, /]$ is meet semi lattice. For $(3, 4)$ there is no LUB $= (\text{Lcm}(3, 4) = 12)$ and 12 is not in set, so not LUB and GLB.

$S = \{ \{a\}, \{b\}, \{a, b\} \}$ then $[S, \subseteq]$ is join semi lattice.

$A = \{1, 2, 3, 4\}$ $[A, \leq]$ is a lattice.

[GLB does not exist for $\{a\}, \{b\}$ (Intersection of $\{a\}, \{b\} = \emptyset$ (not in set).]

$$\begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \quad \begin{array}{c} \text{a, b} = (\max(a, b)) \\ \text{LUB} = (\min(a, b)) \end{array}$$

A TOS (Total order set) is always a lattice.

7. LATTICE EXAMPLES.

→ If A is set of all +ve integers, then poset $[A; /]$ is a lattice.

→ If n is a positive integer then $D_n =$ set of all +ve divisors of n .

Ex: $D_6 = \{1, 2, 3, 6\}$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Every $[D_n, /]$ is a lattice.

→ If $P(A)$ denotes powerset of A then $[P(A), \subseteq]$ is a lattice.

$A = \{a, b\}$

$P(A) = \{ \emptyset, \{a\}, \{b\}, \{a, b\} \}$

$\{a, b\}$

8. PROPERTIES OF LATTICE

The following property holds good in a lattice for any 3 ele $a, b, c \in A$

i) Commutative law: $a \vee b = b \vee a$
 $a \wedge b = b \wedge a$

ii) Associative law:

$$(a \vee b) \vee c = a \vee (b \vee c)$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

iii) Idempotent law: $a \vee a = a$
 $a \wedge a = a$

(iv) Absorption: $a \vee (a \wedge b) = a$
 $a \wedge (a \vee b) = a$

v) Note: In a lattice $(a \vee b) = b$ iff $(a \wedge b) = a, \forall a, b \in L$

$$\begin{cases} V = \text{LUB} \\ \wedge = \text{GLB} \end{cases}$$

9. DISTRIBUTIVE LATTICE AND SUBLATTICE

The lattice on which the distributive property holds is called Distributive lattice. $\Rightarrow$ Every finite lattice, and the distributive properties are

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
 ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

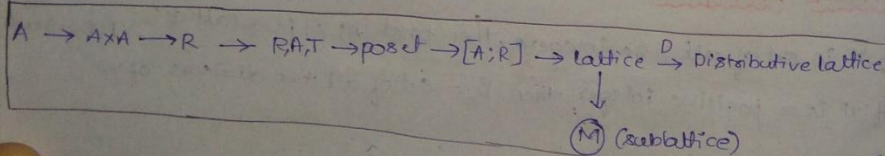
$$\begin{cases} V = \text{LUB} \\ \wedge = \text{GLB} \end{cases}$$

Sublattice:

Let L be a lattice $[L, \wedge, \vee]$. A subset 'M' of 'L' is called a sublattice of 'L' iff

- i) M is a lattice i.e. $[M, \wedge, \vee]$
- ii) For any pair of elements $a, b \in M$ the LUB and GLB are same in M and L

To find whether a lattice is distributive there are 2 ways
 $\Rightarrow$ Take all possible Triplets and check distributive properties (Headache)
 $\Rightarrow$ Use the process of / Concept of sublattices



10. BOUNDED

Let 'L' be a such that (a

$\Rightarrow$ similarly if '0' is called

$\Rightarrow$ In a lattice Bounded to

$$\{1, 2, 3, 4\}, \leq\}$$

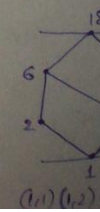
$\Rightarrow$ For a lattice

$$[\mathbb{Z}, \leq]$$

Set of Integers

$$[D_{18}, \mid]$$

$$D_{18} = \{1, 2, 3, 6, 9, 18\}$$



BOUNDED LATTICE

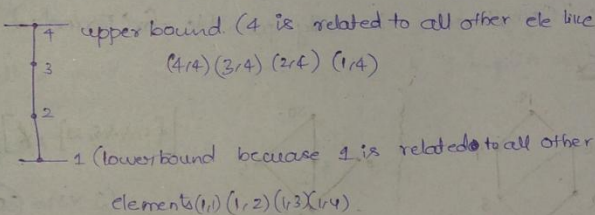
- Let L be a lattice with respect to P , if there exists an element $I \in L$ such that $(a, I) \forall a \in L$, then I is called "Upper bound of Lattice L ".
- Similarly if there exists an element $O \in L$, such that $(O, a) \forall a \in L$, then O is called "Lower bound of Lattice L ".
- In a lattice if upper bound and lower bound exists then it is called Bounded lattice.

LUB, GLB $\rightarrow$ Calculated on pair of elements.

Upper Bound } $\rightarrow$ Calculated on entire lattice not on pair of elements
Lower Bound }

Distributive $\Rightarrow$ Every finite lattice is Bounded.

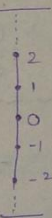
$$\{1, 2, 3, 4\}, \leq$$



$\Rightarrow$ for a lattice there may/maynot be upper bound and lower bound.

$$[I, \leq]$$

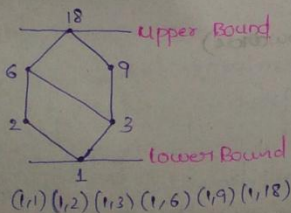
Set of Integers.



No upper bound and lower bound.

$$\Rightarrow [D_8, \mid]$$

$$D_8 = \{1, 2, 3, 6, 9, 18\}$$



11. PROPERTIES OF BOUNDED LATTICE

In a Bounded lattice, the following properties hold good.

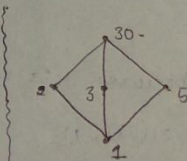
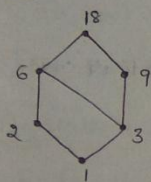
- 1) LUB of a and I is $a \vee I = I$ $\{ (a, I) \}$ $\left\{ \begin{array}{l} I = \text{Upper bound of lattice} \\ 0 = \text{Lower bound of lattice} \end{array} \right.$
- 2) GLB of a and I is $a \wedge I = a$ $\{ (a, I) \}$
- 3) LUB of a and 0 is $a \vee 0 = a$ $\{ (0, a) \}$
- 4) GLB of a and 0 is $a \wedge 0 = 0$ $\{ (0, a) \}$

12. COMPLIMENT OF AN ELEMENT

Let L be an bounded lattice, for any element $a \in L$, if there exists an element $b \in L$ such that $(a \vee b) = I$ and $(a \wedge b) = 0$, then b is called 'Compliment of a ' written as $\bar{a}$. and a, b are compliments of each other.

⇒ "Compliment" is only possible for "Bounded lattice."

Ex:



$$\{1, 2, 3, 5, 30\}$$

$$(2 \vee 3) = 30 \quad (\text{Join operation})$$

$$(2 \wedge 3) = 1 \quad (\text{Meet operation})$$

⇒ $(2, 3)$ are complement of each other ⇒ $\bar{2} = 3$ or $\bar{3} = 2$

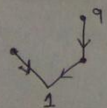
$(2, 9)$ are complement to each other

$$\Rightarrow (2 \vee 9) = \text{Join operation} = 18 \quad (\text{UB})$$

$$(2 \wedge 9) = \text{meet operation} = 1 \quad (\text{LB})$$

$$2 = \bar{9} \text{ or } 9 = \bar{2}$$

$(2, 9)$ are complement



⇒ meet operation (come downwards)



= Join operation (go upwards)

⇒ $(2, 3)$ are not complement to each other because

$$(2 \vee 3) = 6 \quad (\text{Not UB})$$

$$(2 \wedge 3) = 1 \quad (\text{LB})$$

we should get both UB & LB.

13. COMPLEMENT

⇒ of every lattice.

⇒ In a complement

⇒ In a Distributive each element

$$\{1, 2, 3, 6\}$$

$$\{1, 2, 3, 5, 30\}$$

⇒ In a Distributive

14. BOOLEAN ALGEBRA

Boolean Algebra

A lattice L is complemented

⇒ In Boolean Algebra

Unique complement

15. MAXIMAL

Maximal element:

other element, then

Minimal element:

It is called minimal

Ex: $A = \{a, b\}$

$$[P(A), \subseteq]$$

$$\{a\}$$

13

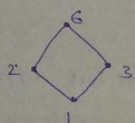
13. COMPLEMENTED LATTICE

⇒ If every element of a lattice has complement then it is called Complemented lattice.

⇒ In a complemented lattice, each element has at least one complement.

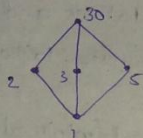
⇒ In a distributive lattice, complement of an element if exists, is unique i.e. each element has at most one complement.

$$[1, 2, 3, 6; 1]$$



→ Complemented lattice every ele has complement.

$$[1, 2, 3, 5, 30; 1]$$



→ Complement lattice

⇒ In a distributive lattice each ele has (0 complement & 1 complement).

14. BOOLEAN ALGEBRA

Boolean Algebra

A lattice 'L' is called Boolean Algebra if it is distributive and complemented.

⇒ In Boolean Algebra every element has at most one complement. i.e. unique complement.

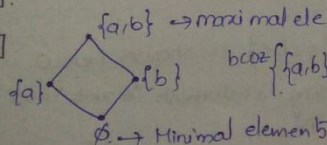
15. MAXIMAL AND MINIMAL ELEMENTS

Maximal element: If in a poset, an element is not related to any other element, then it is called maximal element.

Minimal element: If in a poset, an ele is related to an element, then it is called minimal element.

$$\text{Ex: } A = \{a, b\}$$

$$[P(A); \subseteq]$$



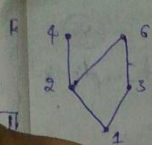
$(\emptyset, \{a\}) \rightarrow \emptyset$ is related to $\{a\}$, Not

$\{a\}$ is related to $\emptyset$.

$\{a, b\}$ is not related to any other element

$\emptyset \rightarrow$ Minimal elements

$\{-, \emptyset\} \rightarrow$ No ele is related to $\emptyset$ so $\emptyset$ is the minimal ele.

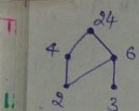


$[1, 2, 3, 4, 6]; 1]$

Maximal elements = 4, 6.
Minimal elements = 1

This is not a lattice because $\{4, 6\}$ do not have a LUB. because and it is Meet semi lattice.

(36)



Maximal elements = 4, 6.
Minimal elements = 2, 3.

Join semi lattice

The upper Bound and lower bound are unique but maximal/minimal elements need not be unique.

⇒ Maximal, Minimal are applicable to both lattices and posets but UB, LB can be/must be applied only for Bounded lattices.

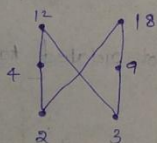
⇒ In a poset if you have more than one maximal/minimal element then it won't be lattice.

16. Example 1

The poset $[2, 3, 4, 9, 12, 18]; 1]$ is

- Join semi lattice but not meet semi lattice
- Meet semi lattice but not Join SL
- A lattice
- Neither join nor meet semi lattice.

⇒



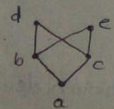
Maximal element = 12, 18
Minimal element = 2, 3

∴ This is not lattice

And there is no GLB for $\{12, 18\}$ and $\{9, 12\}$. They are not LUB for 12, 18. Join and meet SL

17. Example-2

The poset diagram of a poset $P = \{a, b, c, d, e\}$ is shown below.



which of the following statements is not TRUE (FALSE)?

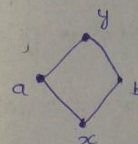
- P is not lattice (True)
- The subset $\{a, b, c\}$ is lattice (T)
- The subset $\{b, c, d, e\}$ is lattice (F)
- The subset $\{a, b, c, e\}$ of P is lattice (T)

18. Example-3

The Hasse diagram which of the following

- $\{x, a, b, y\}$
- $\{x, a, c, y\}$
- $\{x, c, d, y\}$

- $\{x, a, b, y\}$



⇒ lattice because every pair of elements has a LUB, GLB exists.

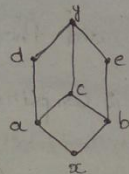
⇒ For sublattice for LUB of every pair the options, Here, GLB and LUB exist but in original diagram LUB $(a, b) = c$ (≠) cannot be the so

- a) TRUE because (d,e) do not have least upper bound.
 b) TRUE all pairs have LUB, GLB
 c) FALSE (d,e) has no LUB and (b,c) do not have GLB

18. Example-3

The Hasse diagram of a lattice $L = \{x, a, b, c, d, e, y\}$ is shown below which of the following subsets of L are sublattices of L ?

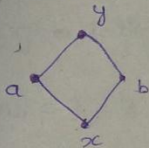
- a) $\{x, a, b, y\}$ ~~b) $\{x, a, c, y\}$~~
 c) $\{x, a, e, y\}$ ~~d) $\{x, d, e, y\}$~~
 e) $\{x, c, d, y\}$



GLB = $\emptyset$

LUB = $\{a, b\}$

- a) $\{x, a, b, y\}$



$\Rightarrow$ Lattice because for every pair of elements both GLB and LUB exists.

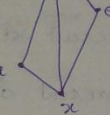
$\Rightarrow$ Lattice bcoz LUB, GLB exists for every pair

$\Rightarrow$ For sublattice find the GLB, LUB of every pair of elements in the options. Here, for (a,b) the GLB and LUB are x,y respectively

but in original diagram the LUB (a,b) = c ($\neq y$). $\therefore$ This cannot be the sublattice of L .

These are same in both diagrams $\therefore$ It is sublattice

For (a,c) = GLB = c
LUB = a
For (c,y) = GLB = c
LUB = y

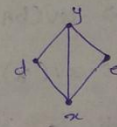


$\Rightarrow$ Lattice for (a,c) GLB = x
LUB = y } Same in original diagram

For (a,y) GLB = a
LUB = y

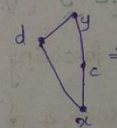
$\therefore$ This is the sublattice

- d) $\{x, d, e, y\}$



$\Rightarrow$ Lattice
 $\Rightarrow$ Sublattice also

- e) $\{x, c, d, y\}$

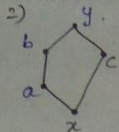
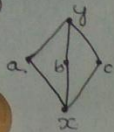


$\begin{cases} \text{GLB}(d,c) = x \\ \text{LUB}(d,c) = a \end{cases}$
 $\downarrow$
 (In original diagram)

$\therefore$ Not a sublattice

19. EXAMPLE-4 (V. Imp Question)

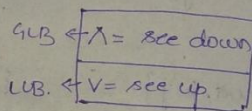
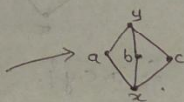
Which of the following lattices is not distributive



$\Rightarrow$ If a lattice is distributive then we should have atmost one complement for each element. In the first diagram 'a' has two complements 'b' and 'c' and therefore it cannot be distributive.

$\Rightarrow$ In the 2nd Graph 'c' has 2 complements 'a' and 'b'. So the 2nd diagram is also not distributive.

Let L_1^* represent diag 1
 L_2^* represent diag 2.



$\Rightarrow$ In $L_1^* \Rightarrow a \vee (b \wedge c) \stackrel{?}{=} (a \vee b) \wedge (a \vee c)$

$$\begin{aligned} &\Rightarrow a \vee (x) && (y) \wedge (y) \\ &\Rightarrow a && = y \end{aligned}$$

$\Rightarrow (a \neq y) \therefore$ The distributive property doesnot hold true

$\Rightarrow$ In $L_2^* \Rightarrow a \vee (b \wedge c) \stackrel{?}{=} (a \vee b) \wedge (a \vee c)$

$$\begin{aligned} &\Rightarrow a \vee (x) && \Rightarrow (b) \wedge (y) \\ &\Rightarrow a && \Rightarrow b \end{aligned}$$

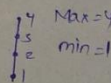
$(a \neq b) \Rightarrow$ The distributive property doesnot hold true for this lattice

20. EXAMPLE-5 (V. Imp Question)

Which of the following statements are not true

a) A lattice with 4 or fewer elements is distributive (TRUE)

b) Every totally ordered set is a distributive lattice (TRUE) $\rightarrow$



c) Every sublattice of a distributive lattice is also distributive (TRUE) $\rightarrow$

d) Every distributive lattice is a bounded lattice

No isomorphic structures exist similar to L_1^* or L_2^* .

$\Rightarrow$ we have already seen L_1^*, L_2^* as sublattice and they are not distributive lattices. So if a lattice contains L_1^* or L_2^* as sublattice

(38)

then that
d) No Relati

21. EXAMPLE

Which of the

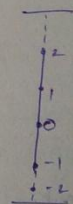
- a) $[P(A), \subseteq]$
- b) $[D_8, \mid]$
- c) $[R; \subseteq] R$
- d) $[\{1, 2, 3, 5, 30\}; \mid]$

$\Rightarrow$ To check wh
 L_2^* are not
the given lat

a) $[P(A), \subseteq]$
union and
we know that
and Intersect
Distributive

b) $D_8 = \{1, 3, 9, 5\}$
For every val

c) $[R; \subseteq]$



d) $[\{1, 2, 3, 5, 30\}; \mid]$

~~D_{30}~~

38

then that lattice won't be ~~sublattice~~ distributive lattice.

39 No Relation b/w Distributive and Bounded lattices (FALSE)

Example-6

Which of the following is not distributive lattice?

- a) $[P(A), \subseteq]$ where $A = \{a, b, c, d\}$.
- b) $[D_8, \mid]$
- c) $[R, \leq]$ R is set of Real numbers
- d) $[1, 2, 3, 5, 30; \mid]$

To check whether a lattice is distributive or not we check if L_1^* and L_2^* are ~~not~~ sublattices of given lattice, if they are ~~sublattices~~ then the given lattice is not distributive.

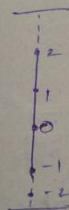
a) $[P(A), \subseteq]$ $A = \{a, b, c, d\}$ w.k.t the join of two elements is nothing but union and meet of two elements is nothing but Intersection and we know that on a set of all sets Union is distributive over Intersection and Intersection is distributive over union. Therefore the option a is Distributive

b) $D_8 = [1, 3, 9, 27, 81; \mid]$

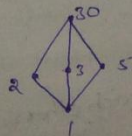
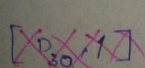
For every value of 'n' $[D_n; \mid]$ is always Distributive

c) $[R; \leq]$

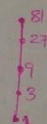
⇒ This is a Total order Relation and w.k.t every Total order Relation is Distributive.



d) $[1, 2, 3, 5, 30; \mid]$



= Isomorphic to L_2^* so this lattice is Not distributive. and $(2, 3)(5, 30)$ are Complements. '2' has two Complements.



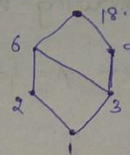
increasing
the
sequence

22. Example-7

For the lattice $[D_8; 1]$ which of the following is not True.

- The complement of $1 = 18$
- The complement of $2 = 9$
- The complement of $3 = 6$ ($3 = \bar{2}$)
- The complement of '6' does not exist

$$D_8 = \{1, 2, 3, 6, 9, 18\}$$



$$(6 \vee 9) = 18 \in \text{LUB}$$

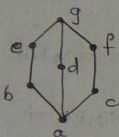
$$(6 \wedge 9) = 3 \text{ (Not GLB)}$$

$\{GLB = 18 \text{ in the lattice}\}$

6 & 9 are not complement

23. Example-8

For the lattice given below, how many complements does the ele 'e' have?



(e, a)

(e, b)

(e, c)

(e, d)

(e, f)

(e, g)

These are the combinations possible

Now, The upper bound of lattice = g

lower bound of lattice = a

Join = $\wedge$ = lower bound

meet = $\vee$ = upper bound

$$\begin{aligned} (e \wedge a) &= a \\ (e \vee a) &= e \end{aligned} \quad \left. \begin{aligned} (e, a) &\text{Not complement} \end{aligned} \right\}$$

$$\begin{aligned} (e \wedge b) &= b \\ (e \vee b) &= e \end{aligned} \quad \left. \begin{aligned} &\text{Not complement} \end{aligned} \right\}$$

$$\begin{aligned} (e \wedge d) &= a = \text{LB of lattice} \\ (e \vee d) &= g = \text{UB of lattice} \end{aligned} \quad \left. \begin{aligned} (e, d) &\text{are complement} \end{aligned} \right\}$$

$$\begin{aligned} (e \wedge f) &= a \\ (e \vee f) &= g \end{aligned} \quad \left. \begin{aligned} (e, f) &\text{are complement} \end{aligned} \right\}$$

Similarly (e, c) are complement to each other.

1. ALGEBRA

A Non-empty set
Binary operation

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100\}$$

$$* \rightarrow$$

$$(S, *)$$

$$S = \{\emptyset, \{1, 2, 3\}\}$$

$$* = \cup$$

Now (S, *)

$$3) A = \{1, 2, 3\}$$

$R =$ Reflexive
set of all

$$(R, \cup) = A$$

$$(R, \cap) = A$$

$$4) (R, +) \Rightarrow$$

$$5) (N, *) \Rightarrow$$

$$6) (S = \{1, 2, 3\}, *)$$

$$* \rightarrow \text{mult}$$

Now, 2*

$$7) [Z, 1] \Rightarrow$$

4. Groups.

1. ALGEBRAIC STRUCTURES

A Non-empty set 'S' is called an Algebraic structure with respect to binary operation * if $(a*b) \in S \forall a, b \in S$ i.e. * is closure operation on 'S'.

1) $S = \{1, -1\}$.

* $\rightarrow$ $\times$ (multiplication)

$(S, *)$ is Algebraic structure because $(1) \times (-1) = -1$ (present in 'S').

$-1 \times -1 = 1$ (e.s.)

$1 \times 1 = 1$ (e.s.)

$\therefore$ 'S' is called the Algebraic Structure.

2) $S = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

* = $\cup$ (union operation)

Now $(S, *)$ is A.S. because

$\emptyset \cup \{a\} = \{a\} \in S.$

$\emptyset \cup \{b\} = \{b\} \in S.$

$\{a\} \cup \{a\} = \{a\} \in S.$

$\emptyset \cup \emptyset = \emptyset \in S.$

$\{a, b\} \cup \{a\} = \{a, b\} \in S.$

3) $A = \{1, 2, 3\}$

$R =$ Reflexive Relations = $\{(1,1), (2,2), (3,3)\}$
set of all

$(R, \cup) = A.S.$

$(R, \cap) = A.S.$

The set of all Reflexive Relations are closed under Intersection and therefore set of all Reflexive Relations w.r.t to Intersection is an "ABELIAN STRUCTURE"

4) $(R, +) \Rightarrow$ "ABELIAN STRUCTURE" (R = Real nos)

5) $(N, *) \Rightarrow$ "ABELIAN STRUCTURE" (N = Natural numbers)

6) $(S = \{1, 2, 3\}, *) \Rightarrow$ (NOT Algebraic STRUCTURE)

* $\rightarrow$ multiplication

Now, $2*3 = 6$ (Not in S). $\therefore$

7) $[2, 1] \Rightarrow$ Not Algebraic structure.

$2/3 = 0.6 \neq \text{Integer}$

2. SEMI GROUP

An Algebraic structure $(S, *)$ is called a Semigroup if $(a * b) * c = a * (b * c) \forall a, b, c \in S$ i.e. $*$ is Associative on 'S'.

Ex:

Natural no = Natural No.

1) $(N, +) = \text{Algebraic Structure } (V) \rightarrow (a+b)+c = a+(b+c)$

2) $(N, *) = \text{Algebraic Structure } (V) \rightarrow (a \times b) \times c = a \times (b \times c) \therefore \text{SEMI GROUP}$

3) $(Z, -) = Z = \{\text{Integers Set}\} \text{ AS } (V) \rightarrow (a-b)-c = a-(b-c) \text{ NOT SEMI GROUP}$

4) $(Q^*, +) = Q = \{\text{Rational Nos. not having '0'}\} \text{ AS } (X) \rightarrow a + (-a) = 0 \text{ NOT RATIONAL}$

5) $(Q^*, *) = \text{Semi Group}$

6) $(P(A), \cup) = \text{Semi Group}$

7) $(P(A), \cap) = \text{Semi Group}$

3. MONOID

A semigroup $(S, *)$ is called Monoid if there exists an element $e \in S$ such that $(a * e) = (e * a) = a \forall a \in S$. ($*$ = operation defined)

The element 'e' is called Identity element of 'S' w.r.t. '*'.

If a semi Group contains Identity element (e) then it is a "Monoid"

Ex:

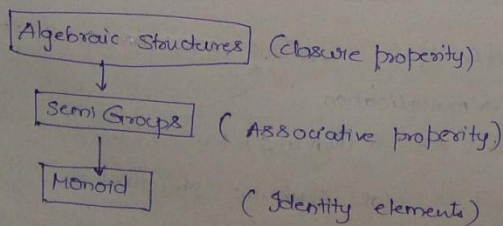
$(N, *) = \text{Semigroup } (V) \rightarrow a * e = a \text{ Now } e = 1 \therefore (N, *) \text{ is Monoid}$

and '1' is the identity element of N w.r.t. '*'

$(N, +) = \text{AS } (V) \text{ SG } (V) \rightarrow a + e = a \Rightarrow e = 0 \notin N \therefore \text{This is not Monoid.}$

$(Z, +) = \text{AS } (V) \text{ SG } (V) \rightarrow a + e = a \Rightarrow e = 0 \in Z \therefore \text{Monoid}$

$(P(A), \cup) = \text{AS } (V) \text{ SG } (V) \rightarrow x \cup e = x \Rightarrow e = \emptyset \in P(A) \therefore \text{Monoid}$



4. GROUP

A monoid $(S, *)$ each element $(b * a) = e$ then

Ex

$(Z, +) = \text{Monoid}$

$(Q, \cdot) = (\text{Multi})$

$(Q^*, \cdot) \Rightarrow Q^* = (\text{Set of no's})$

$\cdot = \text{Multi}$

$(P(A), \cup) = X \cup \{\emptyset\}$

5. ABELIAN GROUP

$\rightarrow$ In a Group

i) The Identity

ii) The Inverse

iii) The Inverse

iv) Cancellation law

v) $(a * b)^{-1} = b^{-1} * a^{-1}$

6. Example 1

Abelian Group (commutative)

A group $(G, *)$ is

Ex:

$(Z, +) = \text{Group } (V) \rightarrow a + (-a) = 0$

$(Q^*, \cdot) = \text{Group } (V) \rightarrow a * a^{-1} = 1$

$(M, *) = m = (\text{Set of all } m)$

42
 $a * b * c$
 Natural No.
 $a + (b + c)$
 SEMI GROUP
 IT SEMI GROUP

4. GROUP

A monoid $(S, *)$ with identity element 'e' is called a group if to each element $a \in S$, there exists an element $b \in S$, such that $(a * b) = (b * a) = e$ then 'b' is called Inverse of an element a, denoted by a^{-1} .

Ex

$(\mathbb{Z}, +) = \text{Monoid}(\mathbb{Z})$
 $= a + b = b + a = 0 \text{ (Identity element)}$
 $= \boxed{a = -b}$ and $(-b) \in \mathbb{Z} \therefore \text{Group}$ | For every ele there is Inverse

$(\mathbb{Q}, \cdot) = (\text{Multiplication}) = a * b = 1 \therefore \text{This is a Group}$
 $\Rightarrow \boxed{b = \frac{1}{a}} \notin \mathbb{Q} \text{ if } (a = 0)$ Not

$(\mathbb{Q}^*, \cdot) \Rightarrow \mathbb{Q}^* = (\text{set of all Rational nos without } 0)$
 $a * b = 1 \Rightarrow \boxed{a = \frac{1}{b}} \quad \boxed{b = \frac{1}{a}} \in \mathbb{Q}^*$
 $\therefore \text{Multiplication} \Rightarrow \text{This is a Group because the set } \mathbb{Q}^* \text{ does not has } 0.$

$(\mathbb{P}(A), \cup) = X \cup \{\phi\} = \{\phi\} \Rightarrow \text{identity element}$
 $\downarrow$
 $\boxed{\phi \in \mathbb{P}(A)}$
 $\therefore \text{Group}$

"Monoid"

5. ABELIAN GROUP

$\rightarrow$ In a Group $(G, *)$ the following properties must hold good

- i) The Identity element of 'G' is unique
- ii) The Inverse of any element in 'G' is unique
- iii) The Inverse of identity element 'e' is 'e' itself
- iv) Cancellation laws $(a * b) = (a * c) \Rightarrow b = c$
 $(a * c) = (b * c) \Rightarrow a = b$

v) $(a * b)^{-1} = b^{-1} * a^{-1} \quad \forall a, b \in G$

6. EXAMPLE 1

Abelian Group (Commutative Group)

A group $(G, *)$ is said to be Abelian if $(a * b) = (b * a) \quad \forall a, b \in G$.

Ex:

$(\mathbb{Z}, +) = \text{Group}(\mathbb{Z}) \quad a + b = b + a \in \mathbb{Z} \Rightarrow \text{Abelian Group}$

$(\mathbb{R}^+, \cdot) = \text{Group}(\mathbb{R}^+) \quad a * b = b * a \in \mathbb{R}^+ \Rightarrow \text{Abelian Group}$

$(M, *) = M = (\text{set of all non singular matrices}) \Rightarrow (* = \text{matrix multiplication}) = \text{NOT Abelian Group}$

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 quence

Example 8

7. EXAMPLE 2

Which of the following is/are True?

1) In a group $(G, *)$ an identity element 'e' if $a * a = a$ then $a = e$ with

2) In a Group $(G, *)$ if $x^{-1} = x \forall x \in G$ then 'G' is Abelian Group.

3) " " " " " $(a * b)^2 = a^2 * b^2 \forall a, b \in G$ then 'G' is Abelian Group.

⇒ Use the formulas in 5th video. Now, $a * a = a$

1) $a * a = a * e$ (I can write 'a' as $a * e$ because 'e' is identity element)
 $a = e$ TRUE.

2) $x^{-1} = x$
 $x^{-1} = x * e$

$$(a * b)^{-1} = b^{-1} * a^{-1}$$

$$(a * b) = (b * a) \Rightarrow \text{Given } x^{-1} = x \text{ so } (a * b)^{-1} = (a * b)$$

$b^{-1} = b$
 $a^{-1} = a$

3) $a^2 = a * a$ write the operation specified.

$$\Rightarrow (a * b)^2 = a^2 * b^2$$

$$= (a * b) * (a * b) \quad (a * b) * (a * b) = a * a * b * b$$

$$= (a * a) * (b * b) = (a * a) * (b * b) = \text{LHS} = \text{RHS}$$

$a * (b * a) * b =$
 $a * a * b * b$
 $(b * a) = (a * b)$

8. EXAMPLE - 3

If $A = \{1, 3, 5, 7, 9, \dots, \infty\}$ and $B = \{2, 4, 6, 8, \dots, \infty\}$ which of the following is semigroup?

a) $(A, +)$ = Algebraic structure (X)

b) $(A, \cdot)$ = AS (✓) SGr (✓) Monoid (✓) ($e = 1$)

c) $(B, +)$ = AS (✓) SGr (✓)

d) $(B, \cdot)$ = AS (✓) SGr (✓)

$$a * \{2\} = 1$$

$a = 1/2 \notin A \therefore$ Not Abelian.

9. EXAMPLE

Let $A = \{1, 2, 3, \dots\}$

$\forall a, b \in A$ which

a) $(A, *)$ is SGr

b) $(A, *)$ is n

c) $(A, *)$ is G

d) $(A, *)$ is no

10. EXAMPLE

Let $A = \{x / \infty\}$

is

a) A semigroup

b) A monoid

c) A group

d) Not a semigroup

$(A, *) \Rightarrow AS$

11. EXAMPLE -

Let 'A' is set

$a * b = \min(a, b)$

$(\mathbb{Z}, \cdot) \Rightarrow$

9. EXAMPLE-4

Let $A = \{1, 2, 3, 4, \dots, \infty\}$ and a Binary operation $*$ is defined by $a * b = a^b$ $\forall a, b \in A$ which of the following is true?

- a) $(A, *)$ is Semi Group but not monoid
- b) $(A, *)$ is monoid but not group
- c) $(A, *)$ is Group
- d) $(A, *)$ is not semi Group.

$$a * (b * c) = (a * b) * c$$

$$1 * (2 * 3) = (1 * 2) * 3$$

$$= 1 * (2^3) = (1^2) * 3$$

$$= 8 \quad \quad = (1 * 3) = 1^3$$

$$= 1$$

$$(8 \neq 1)$$

so not SG

10. EXAMPLE-5

Let $A = \{x / 0 < x \leq 1 \text{ and } x \text{ is a Real number}\}$ then 'A' w.r.t to multiplication is

- a) A semigroup but not monoid
- b) A monoid but not Group
- c) A group
- d) Not a semigroup

$(A, *) \Rightarrow$ AS(V) Semi Group (✓)

$$(a * b) * c = a * (b * c)$$

$$\downarrow$$

$$0.1 * 0.1 = 0.1 * 0.1$$

$$= (0.1)^2 = (0.1)^2$$

Monoid (✓)

$$e = 1$$

Identity element

Group.

$$a * \{x\} = e$$

$$\Rightarrow x = \frac{1}{a} \notin A$$

$\frac{1}{0.1} = 10 \notin A$

11. EXAMPLE-6

Let 'A' is set of all integers and Binary operation $*$ is defined by

$(a * b) = \min(a, b)$ then $(A, *)$ is.

$(\mathbb{Z}, *) \Rightarrow$ Algebraic Structure (✓)

Semi Group (✓)

$$(a * b) * c = a * (b * c)$$

$$\downarrow$$

$$1 * 2 * 3 = 1 * (2 * 3)$$

$$= 1 * 3 = 1$$

Monoid

$$a * \{e\} = a$$

$$\downarrow$$

$$a * e = \min(a, e)$$

$$\downarrow$$

$$101 * 100 = 100 \neq 101$$

Group.

$$a * b = e \Rightarrow a * ? = e$$

$$\Rightarrow a * ? = 1$$

$$\Rightarrow ? = \frac{1}{a} \notin \mathbb{Z}$$

Semi Group.

It is Monoid but not Group

16. FINITE GROUPS.

A Group with finite no. of elements is called finite Group.

$O(G) \rightarrow$ order of finite Group. (No. of ele in the Group)

Ex:

1) $(\{0\}, +) = (0+0)=0$, Identity element = '0'. (Monoid) Group (V)

$\Rightarrow$ whenever a group is having only one element then that ele will be the Identity element. (V.V. Imp.)

2) $(\{1\}, *) = 1*1 = 1 \in \text{set } (AS) \checkmark$

$(1*1)*1 = 1*(1*1) (SG) \checkmark$

$a*e = a = 1*a = 1 \Rightarrow e=1 \in \text{set (monoid)} \checkmark$

$1*a*x = e \Rightarrow x = 1/a = 1/1 \Rightarrow x=1=e \therefore \text{(Group)}$

3) $(\{1, -1\}, *) = \text{Group } (V)$

= composition table

	1	-1
1	1	-1
-1	-1	1

$\in \text{set} \therefore (AS) \checkmark$

= (SG) $\checkmark$

= Monoid $\checkmark$ ($e=1$ Identity element)

= Group.

4) $(\{1, \omega, \omega^2\}, *) =$

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

AS(V)
Identity ele = 1 (mon) $\checkmark$
(SG) $\checkmark$

$\omega^4 = \omega^3 \times \omega$
 $= 1 \times \omega$
 $\omega^4 = \omega$

For all the ele we are able to find inverses $\therefore$ Group

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

ω is the Inverse of ω^2
 ω^2 is the Inverse of ω
 ω is the Inverse of ω

18. EXAMP

If $G = \{1, 3, 5, 7\}$

a) The inv

b) The Inv

Now, 1x

3x

5x

7x

$\therefore$ Ea

2) which of

a) $\{1, 2, 3, 4\}$

b) $\{0, 1, 2, 3, 4\}$

c) $\{1, 2, 3, 4, 5\}$

d) $\{1, 2, 3, 4, 5\}$

a) $\{1, 2, 3, 4, 5\}$

= (2x

$\Rightarrow$ C

$\Rightarrow$ (A

c) $\{1, 2, 3, 4, 5\}$

S_7 (x

$\therefore$ Group.

EXAMPLES ON FINITE GROUPS

If $G = \{1, 3, 5, 7\}$ is a group w.r.t $\times_8$, which of the following is not true?

(a) The inv of 1 is 1

(b) The inverse of 5 is 7

(c) The inv of 3 is 3

(d) The inverse of 7 is 7.

active 1/3

(27)

Now, $1 \times 1 = 1$ (1 when divided by 8 gives '1' as Remainder)

$3 \times 3 = 9$ ($9 \div 8 = 1$ (belong to G) and 1 is identity element)

$5 \times 5 = 25$ ($25 \div 8 = 1$ (identity element))

$7 \times 7 = 49$ ($49 \div 8 = 1$ (identity element))

$\therefore$ Each element is Inverse of itself

(a)

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

1 is the Inverse of 1

3 " " " 3

5 " " " 5

7 " " " 7

How did I write this? $(7 \times 5) \div 8 = 35 \div 8 = 3$

2) which of the following is a Group?

(a) $\{1, 2, 3, 4\}$ w.r.t $\times_6$

we know that

(b) $\{0, 1, 2, 3, 4, 5\}$ w.r.t $\times_6$

$\oplus_m$ contains $0, 1, 2, \dots, (m-1)$ elements

(c) $\{1, 2, 3, 4, 5, 6\}$ w.r.t $\times_7$

$\otimes_m$ contains 'm' elements.

(d) $\{1, 2, 3, 4, 5, 6\}$ w.r.t $\times_7$

(a) $\{1, 2, 3, 4, 5\}$ w.r.t $\times_6$

(b) $\{0, 1, 2, 3, 4, 5\}$ w.r.t $\times_6$

$$= (2 \times 3) \div (6k)$$

$\Rightarrow 0 \notin \text{set}$

$\Rightarrow (AS) \times$

$\Rightarrow$ No Inverse to '0' and we cannot get

Identity element so doesn't form Group.

(d) $\{1, 2, 3, 4, 5, 6\}$ w.r.t $\oplus_7$

(0, 6) must be present

'0' is missing.

Monoid property fails

(c) $\{1, 2, 3, 4, 5, 6\}$ w.r.t $\times_7$

S_7 (set of all nbs less

than 7 and are

relatively prime to 7).

Group.

18. EXAMPLES ON FINITE GROUPS

1) $(\{0, 1, 2, \dots, (m-1)\}, \oplus_m)$ Addition modulo m (This will always be Group)

$a \oplus_m b = \begin{cases} a+b < m \Rightarrow (a+b) \text{ is the result} \\ a+b > m \Rightarrow (a+b) \% m \text{ is the result} \end{cases}$

$(\{0, 1, 2\}, \oplus_3) \Rightarrow$ This is a group

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$(0, 1, 2) \in \text{set} \therefore \text{As } \checkmark (S_6) \checkmark (m) \checkmark (G) \checkmark$

$\Rightarrow$ for getting the identity element check the row containing $\{0, 1, 2\}$ (in the same order) then that element will be Identity element here '0' row has $\{0, 1, 2\} \therefore$ '0' is the identity element

$\Rightarrow$ Here in this case check for $a \cdot b$

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\Rightarrow 2$ is the Inverse of 1
 $\hookrightarrow 1$ is the Inverse of 2.

$$a \cdot a^{-1} = e$$

$$a \cdot a^{-1} = 0$$

$$a^{-1} = 0$$

2) $(S_m, \otimes_m) \Rightarrow a \otimes_m b = (a \cdot b) \% m$

$\downarrow$

Set of all nos that are less than m and relatively prime to m .

$S_{10} = \{1, 3, 7, 9\} \therefore S_0 = \{1, 3, 7, 9\}$

Two nos are said to be Relatively prime if $\text{GCD}(a, b) = 1$

Now, $\{S_0, \otimes_{10}\} = \text{Group}$

$= \{S_{10}, \otimes_{10}\} = \{\{1, 3, 7, 9\}, \otimes_{10}\} = \text{Group}$

19. ORDER

Order of an
 Let $(G, *)$ be
 smallest pos

Ex:

1) $(\{1, -1\}, \cdot)$

Now, $($

$\Rightarrow$ order of

2) $(\{1, \omega, \omega^2\}, \cdot)$

$1^1 = 1$

$\omega^3 = 1$

$\omega^2 \cdot \omega = 1$

$\therefore \{0, 1\}$

$\Rightarrow$ Order of

3) $(\{0, 1, 2\}, \oplus_3)$

Now, $0^1 =$

$1 \oplus_3 1 =$

$2 \oplus_3 2 =$

In a Gr

ORDER

Order of an element of a group

Let $(G, *)$ be a Group and $a \in G$, then order of element 'a' is the smallest positive integer 'n' such that a^n is identity element.

Ex:

1) $(\{1, -1\}, *)$ Identity element = 1

Now, $(-1)^2 = 1 = \text{identity element} \therefore \text{order of } (-1) = 2$

$(-1)^4 = 1$ '4' cannot be order we should always take least no.

$\Rightarrow$ order of Identity element is always one.

2) $(\{1, \omega, \omega^2\}, *)$ Identity element = 1

$1^1 = 1 \Rightarrow \text{order of } 1 = 1$

$\omega^3 = 1 \Rightarrow \text{order of } \omega = 3$

$\omega^2 \times \omega^2 \times \omega^2 = 1 \therefore \text{order of } \omega^2 = 3$ Now, $(\omega^2)^3 = \omega^6 = (\omega^3)^2 = 1$

$$\therefore \boxed{O(\omega^2) = 3 \quad O(\omega) = 3 \quad O(1) = 1}$$

$\Rightarrow$ Order of any element divides the order of group (Finite Group)

3) $(\{0, 1, 2\}, \oplus_3)$ Identity element = 0

Now, $0^1 = 0 \therefore O(0) = 1$

$$1 \oplus_3 1 \oplus_3 1 = 0 \Rightarrow \boxed{O(1) = 3} \Rightarrow \boxed{1^3 = 0}$$

$$2 \oplus_3 2 \oplus_3 2 = 0 \Rightarrow \boxed{O(2) = 3} \Rightarrow \boxed{2^3 = 0}$$

In a Group $O(a)$ and $O(a^{-1})$ are always same/Equal.

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in the
sequence

20. EXAMPLES ON ORDER

state True/false

1) In a group $(\mathbb{Z}, +)$ the order of any element except '0' does not exist

2) In the group $(\mathbb{Q}^*, \cdot)$ where $\mathbb{Q}^*$ is set of all Non-zero rational numbers,

i.e. $\mathbb{Q}^* = \mathbb{Q} - \{0\}$, the order of any element except 1 does not exist

3) TRUE, because order of '0' is 1 (because '0' is Identity element) and for other ele order does not exist

4) 1 is the Identity element so $O(1) = 1$

Now, $(-1)^2 = 1 \therefore O(-1) = 2 \therefore S_2$ is FALSE.

21. SUBGROUPS.

SUBGROUPS.

Let $(G, *)$ be a group. A subset 'H' of 'G' is called a subgroup of 'G' if $(H, *)$ is a group.

Ex:

Let $(G, *)$ be a group with Identity element 'e', then $\{e\}$ and 'G' are the trivial subgroups of 'G'. Any subgroup which is not a trivial subgroup is called proper subgroup.

$G = (\{1, -1, i, -i\}, *)$ then $H = (\{1, -1\}, *)$ is a proper subgroup.

⇒ Every Group is going to have atleast two subgroups 1) itself
2) set containing Identity elements. (These are called Trivial subgroups)

The subgroups other than Trivial subgroups are called proper subgroups.

22. THEO

Th 1: Let
if $a * b^{-1} \in H$

Th 2: Let 'H' of 'G' if $($

Th 3: Lagrange

if 'H' is of $O(G)$.

$O(G) = m$

$O(H) = n$

23. Examp

Let $G = (\mathbb{Z}_6, +)$
subgroups of

a) $H_1 = \{1, 3\}$

b) $H_2 = \{1, 5\}$

c) $H_3 = \{1, 2, 3\}$

d) $H_4 = \{0, 2, 4\}$

e) $H_5 = \{0, 2, 3\}$

THEOREMS ON SUB GROUPS.

Th 1: Let 'H' be non empty subset of a group $(G, *)$. 'H' is a ^{sub}group of 'G' iff $a * b^{-1} \in H \quad \forall a, b \in H$

Th 2: Let 'H' be non empty finite subset of a group $(G, *)$. 'H' is a subgroup of 'G' iff $(a * b) \in H \quad \forall a, b \in H$

Th 3: Lagrange's Theorem :-

If 'H' is a subgroup of finite group $(G, *)$ then $O(H)$ is the divisor of $O(G)$. The converse of the above theorem need not be True.

$\left. \begin{matrix} O(G) = m \\ O(H) = n \end{matrix} \right\} \Rightarrow m \text{ is divisible with 'n'}. \text{ (This is what this theorem says).}$ Ex: $O(G) = 10$
 $O(H) = 3$ $\rightarrow$ 'H' cannot be subgroup of 'G' because 3 does not divide 10.

13. EXAMPLES ON SUBGROUPS - 1

Let $G = (\{0, 1, 2, 3, 4, 5\}, \oplus_6)$ is a group. which of the following is/are subgroups of G?

1) $H_1 = \{1, 3\} \Rightarrow \text{Now, } \{1, 3\} \oplus_6 = (1+3) \oplus_6 = 4 \bmod 6 = 4 \neq H_1$

2) $H_2 = \{1, 5\} \Rightarrow \text{Now, } \{1, 5\} \oplus_6 = (1+5) \bmod 6 = 0 \neq H_2$

3) $H_3 = \{1, 3\} \Rightarrow \text{Now, } \{1, 3\} \oplus_6 = (4) \bmod 6 = 4 \neq H_3$

4) $H_4 = \{0, 2, 4\} \Rightarrow \text{Now, } \{0, 2, 4\} \oplus_6 = \left. \begin{matrix} 2 \bmod 6 = 2 & (0+2) \bmod 6 = 2 \\ 0 \bmod 6 = 0 & (0+4) \bmod 6 = 4 \\ 4 \bmod 6 = 4 & (2+4) \bmod 6 = 0 \end{matrix} \right\} \in H_4$

5) $H_5 = \{0, 2, 3, 5\} \Rightarrow \text{Now, } \{0, 2, 3, 5\} \oplus_6$

$$\begin{aligned} &= 0 \bmod 6 = 0 \\ &= 2 \bmod 6 = 2 \bmod 6 \\ &= 3 \bmod 6 = 3 \bmod 6 \\ &= 5 \bmod 6 = 5 \bmod 6 = 2, 3, 5 \text{ respectively} \\ &\Rightarrow 0, 2, 3, 5 \in H_5 \end{aligned} \quad \left| \begin{array}{l} (0, 2) \bmod 6 = 2 \\ (0, 3) \bmod 6 = 3 \\ (0, 5) \bmod 6 = 5 \\ (2, 3) \bmod 6 = 5 \\ (2, 5) \bmod 6 = 1 \notin H_5 \end{array} \right.$$

This won't be subgroup.

(OR) "PROVE BY CONSTRUCTING COMPOSITION TABLE"

24. EXAMPLES ON SUB GROUPS - 2

$G = (\{1, 2, 3, 4, 5, 6\}, \otimes_6)$ which of the following are subgroups of 'G'.

a) $H_1 = \{1, 6\}$ c) $H_3 = \{1, 3, 5\}$

b) $H_2 = \{1, 2, 4\}$ d) $H_4 = \{1, 2, 3, 5\}$

H_1 :

	1	6
1	1	6
6	6	1

H_1 is subgroup

H_2 :

	1	2	4
1	1	2	4
2	2	4	1
4	4	1	2

H_2 is subgroup.

H_3 :

	1	3	5
1	1	3	5
3	3	2	1
5	5	1	4

$2 \notin H_3$
 $\therefore H_3$ is not sub group.

H_4 :

	1	2	3	5
1	1	2	3	5
2	2	4	6	3
3	3	6	2	1
5	5	3	1	4

$6 \notin H_4 \therefore H_4$ is not subgroup.

25. EXAMPLES ON SUB GROUPS - 3

Let $(G, *)$ be a group of order 'p' where 'p' is prime no., then the no. of proper subgroups of 'G' is — ?

Sol: Given the order of 'G' has prime number = p (1 and p are only factors)

$\therefore$ The subgroups of G has the order '1' and 'p'.

$\Rightarrow$ so 2 subgroups are possible that are the total set $(G, *)$ and the identity element set say $\{e\}$

But these are Trivial subgroups.

$\therefore$ The total no. of subgroups = Total - Trivial subgroups

$$= 2 - (2)$$

$$\boxed{\text{Subgroups} = 0}$$

26. EXAM

Which of the

a) The union

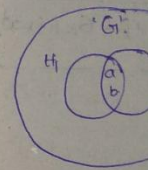
b) The intersect

c) The union

d) Every subg

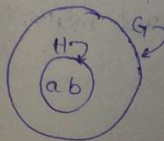
a) $(S_8, \otimes_8)$

b) TRUE. (same



c) FALSE (Refer

d)



27. CYCLIC GROUPS

A group $(G, *)$ is called a cyclic group if there exists an element $a \in G$ such that every element of 'G' can be written as a^n for some integer n . Then 'a' is called generating element/generator.

1) $G_1 = (\{1, -1\}, *)$ Now $(-1)^1 = (-1) = -1$
 $(-1)^2 = 1$ } The elements can be generated using (-1) as $(-1)^1$ and $(-1)^2 = 1$.

$\therefore (-1)$ is the GENERATOR

2) $G_2 = (\{1, \omega, \omega^2\}, *)$ Now, $\omega^1 = \omega$
 $\omega^2 = \omega^2$
 $\omega^3 = 1$ } all the elements can be generated using " ω "

$\therefore$ Generator = ω

3) $G_3 = (\{1, -1, i, -i\}, *)$ Now, $i^1 = i$
 $i^2 = -1$
 $(i^2)^2 = i^4 = 1$
 $i^3 = -i$ } $\therefore$ 'i' is the Generator

4) $G_4 = (\{0, 1, 2, 3\}, \oplus_4)$ Now, $1^1 = 1$
 $1^2 = 1+1 = 2$
 $1^3 = 1+1+1 = 3$
 $1^4 = 0 \quad (4 \bmod 4 = 0)$ } Operation defined here is addition not multiplication

Now, $3^1 = 3$
 $3^2 = 6 \bmod 4 = 2$
 $3^3 = 9 \bmod 4 = 1$
 $3^4 = 12 \bmod 4 = 0$ } '3' is also a Generator

28. EXAMPLES

$\rightarrow$ If $(G, *)$

i) a^{-1} is also

ii) The Ord

Ex: $(\{0, 1, 2, 3\}, +)$

$1^1 = 1 \quad 1^2 =$

	0	1
0	0	1
1	1	2
2	2	3
3	3	4

$3^1 = 3$

$3^2 = 2$

$3^3 = 1$

$3^4 = 0$

$\therefore O(1) = 4$

$O(3) = 4$

Ex $(S_5, \circ) =$

$2^1 = 2, \quad 2^2 = 4$

$2^3 = 3, \quad 2^4 = 4,$

$O(3) = 4$

$O(2) = 4$

EXAMPLES ON CYCLIC GROUPS

→ If $(G, *)$ is a cyclic group with generator 'a' then

i) a^{-1} is also a generator

ii) The Order of the Generator = $O(a)$ = Order of the Group.

Ex: $(\{0, 1, 2, 3\}, \oplus_4)$

$1^1 = 1, 1^2 = 2, 1^3 = 3, 1^4 = 4 \bmod 4 = 0. \therefore 1$ is the generator

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Now, the Inverse of $a * a^{-1} = e$ (Identity ele)

Here 0 is the Identity element $\Rightarrow 1 * 3 = 0$

$\Rightarrow 2 * 2 = 0$

$\therefore$ check in 1 row where '0' is present

$\therefore$ under '3' numbered column.

$\therefore$ Inverse of 1 is '3'

$3^1 = 3$

$3^2 = 2$

$3^3 = 1$

$3^4 = 0$

$\therefore 3$ is also a Generator.

1st point is satisfied

$O(1) = 4$ (because $1^4 = 0$ (Identity element))

$O(3) = 4$ ($3^4 = 0$ (Identity element)).

Ex: $(S_5, \oplus) = (\{1, 2, 3, 4\}, \oplus_5)$

$2^1 = 2, 2^2 = 4, 2^3 = 3, 2^4 = 1$

$3^1 = 3, 3^2 = 4, 3^3 = 2, 3^4 = 1$

2 and 3 are generators

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Now, the identity element is '1' Now

'3' is the Inverse of '2'.

$O(3) = 4$
 $O(2) = 4$
 $\therefore O(2) = O(3)$

29. THEOREM ON CYCLIC GROUPS

Let $(G, *)$ be a cyclic group of order n with generator a then

1) The no. of generators in $G = \phi(n)$ Euler's function of n .

2) a^m is also generator of G if $\text{GCD}(m, n) = 1$

$\phi(n)$ = The no. of numbers that are less than n and relatively prime to n .

Ex

Let $(G, *)$ be a cyclic group of order 8 with generator a

(i) No. of generators in $G = ?$ 4

(ii) which of the following is not a generator of G ?

(a) a^2 (b) a^3 (c) a^5 (d) a^7

Now, Here $n=8 \Rightarrow$ The set $\phi(n) = \{1, 3, 5, 7\}$

$\therefore$ No. of generators in $G = 4$

Now, $n=8$. Now, a^m is also generator if $\text{GCD}(m, n) = 1$

$$\Rightarrow a^2 = \text{GCD}(2, 8) \neq 1 \text{ --- NOT Generator}$$

$$\Rightarrow a^3 = \text{GCD}(3, 8) = 1 \text{ --- Generator}$$

$$\Rightarrow a^5 = \text{GCD}(5, 8) = 1 \text{ --- Generator}$$

$$\Rightarrow a^7 = \text{GCD}(7, 8) = 1 \text{ --- Generator.}$$

The problem with this method is as the value of n increases then it is difficult to find the prime nos less than n . So the product rule has been defined.

If $n = p \times q$ (if n can be written as product of two distinct prime numbers p and q then)

$$\phi(n) = \phi(p) \phi(q)$$

↓

The Advantage with this method is if p is a prime number

$$\text{then } \phi(p) = (p-1) \quad [\phi(7) = 6, \phi(11) = 10]$$

$$\text{Now, } \phi(77) = \phi(7) \times \phi(11)$$

$$= 6 \times 10 = 60$$

Now, So the they are

$$\Rightarrow \text{Now, } \phi(84) \\ \phi(84)$$

$$\phi(84)$$

30. EXAMPLES

$$1) G_1 = \{1, 2, 3, \dots\}$$

$$2) G_2 = \{0, 1, 2, \dots\}$$

$$3) G_3 = \{1, 3, 5, \dots\}$$

$$\text{Now, } G_1 =$$

No. of

Now,

$\therefore$ The 2 generators

Now, In the above procedure Both 'p' and 'q' should be distinct if they are same prime numbers then the formula is

$$\phi(p^n) = p^n - p^{n-1}$$

'p' should be prime

$$\phi(25) = \phi(5^2) = 5^2 - 5$$

$$\phi(25) = 20$$

Now, $\phi(84) = 2 \times 2 \times 3 \times 7$

$$\phi(84) = \phi(2^2 \times 3 \times 7)$$

$$= \phi(2^2) \times \phi(3) \times \phi(7)$$

$$= (2^2 - 2) \times (2) \times (6)$$

$$= 2 \times 2 \times 6$$

$$\phi(84) = 24$$

$$\begin{cases} \phi(3) = (3-1) = 2 \\ \phi(7) = (7-1) = 6 \end{cases}$$

$$\begin{cases} \phi(2^2) = 2^2 - 2^1 \\ = 2 \end{cases}$$

EXAMPLES ON CYCLIC GROUPS:

1) $G_1 = (\{1, 2, 3, 4, 5, 6\}, \otimes_7)$ Find all the generators of G_1, G_2, G_3 .

2) $G_2 = (\{0, 1, 2, 3, 4\}, \oplus_5)$

3) $G_3 = (\{1, 3, 5, 7\}, \otimes_8)$

Now, $G_1 = (\{1, 2, 3, 4, 5, 6\}, \otimes_7)$

No. of generators = $\phi(6) = \{1, 5\} = 2$ generators. $\Rightarrow 1, 5$ are generators

Now, 1 cannot be generator because it is identity element

2 is not generator

$$2^1 = 2$$

$$2^4 = 2$$

$$2^2 = 4$$

$$2^5 = 4$$

$$2^3 = 1$$

$$2^6 = 1$$

$\left. \begin{matrix} 2^1 = 2 & 2^4 = 2 \\ 2^2 = 4 & 2^5 = 4 \\ 2^3 = 1 & 2^6 = 1 \end{matrix} \right\} 4, 5, 6 \text{ are not generated}$

3 is generator.

$$3^1 = 3$$

$$3^5 = 3^2 \times 3^3$$

$$3^2 = 2$$

$$= 2 \times 6$$

$$3^3 = 6$$

$$= 5 \pmod{7}$$

$$3^4 = 3^2 \times 3^2 = 4$$

$$3^6 = 3^3 \times 3^3$$

$$= 6 \times 6$$

$$= 1 \pmod{7}$$

The 2 generators are 3 and 5
= 3 and 5 //

Now,

$$(2) G_2 = (\{0, 1, 2, 3, 4\}, \oplus_5) \quad o(G) = (5)$$

No. of generators = $\phi(5) = \{4\} = \{0, 1, 2, 3, 4\} \Rightarrow a^0, a^1, a^2, a^3, a^4$ are

Now, 0 is the identity element $\Rightarrow$ it cannot be generator

$$1^1 = 1 \quad 1^2 = 2 \quad 1^3 = 3 \quad 1^4 = 4 \quad 1^5 = 0. \quad 1 \text{ is generator.}$$

$$= \begin{matrix} 1^1 & 1^2 & 1^3 & 1^4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{matrix} \text{ are generators.}$$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \text{ are generators}$$

$$(3) G_3 = (\{1, 3, 5, 7\}, \otimes_8)$$

Now, $\phi(4) = \{1, 3\} \therefore a^1, a^3$ are the 2 generators.

Now 1 is identity element

$$3^1 = 3 \quad 3^2 = 1 \quad 3^3 = 3 \quad 3^4 = 1 \therefore 3 \text{ is not generator}$$

$$5^1 = 5 \quad 5^2 = 1 \quad 5^3 = 5 \quad 5^4 = 1 \text{ not generator}$$

$$7^1 = 7 \quad 7^2 = 1 \text{ not generator.}$$

$\therefore$ There are no generators for this group $\Rightarrow$ The group is not cyclic group.

3.1. SOME POINTS ON CYCLIC GROUPS.

For cyclic groups, the following properties hold good. $g = \text{generator (assume)}$

- 1) Every cyclic group is an Abelian Group $[a * b = g^n * g^m = g^{m+n} = g^m * g^n = b * a]$
- 2) Every Group of prime order is cyclic and so every group of prime order is Abelian Group.

- 3) Every subgroup of a cyclic group is also cyclic, but the generator of the subgroup need not be same as that of cyclic group.

Ex: $G = \{1, -1, i, -i\}$; $H = \{1, -1\}$ So, H is subgroup of G

Generators of G are $i, -i$, Generator of $H = -1$

- 4) Let $(G, *)$ be a group of even order, then there exists atleast one element $a \in G$ ($a \neq e$) such that $a^{-1} = a$.

58

1. INTRODU

A Relation to each element denoted as

Range: Ran

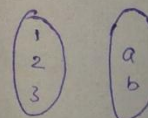
A function

$$f: A \rightarrow B$$

$$\text{Range} = \{1, 2, 3, 4\}$$

2. COUNTING

$$f: A \rightarrow B \quad A \neq \emptyset$$



1 has 2 choices

2 has 2 choices

3 has 2 choices

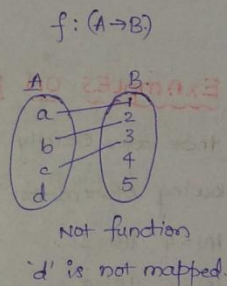
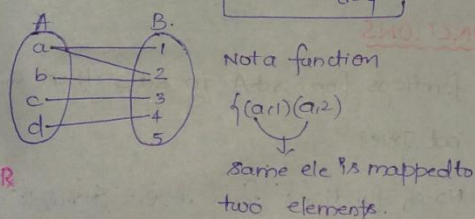
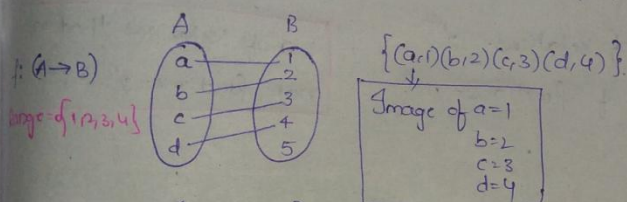
5. FUNCTIONS

INTRODUCTION TO FUNCTIONS

A relation 'f' from 'set A' to a 'set B' is called a function if to each element $a \in A$, we can assign a unique element of 'B'. It is denoted as $f: A \rightarrow B$. A is domain and 'B' is co-domain.

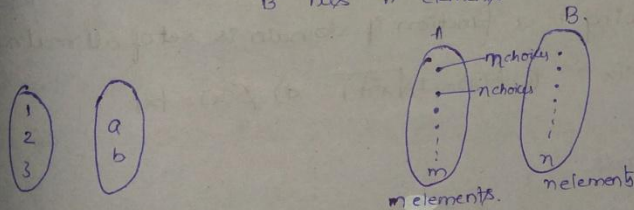
Range: Range of function is $\{y/y \in B \text{ and } (x,y) \in f\}$ so range of f is B.

A function $f: A \rightarrow A$ is called a function on A.



COUNTING THE FUNCTIONS

$f: A \rightarrow B$ Assume 'A' has 'm' elements
 'B' has 'n' elements



1 has 2 choices (1,a)
 (1,b)

2 has 2 choices (2,a) (2,b)

3 has 2 choices (3,a) (3,b)

$n \times n \times n \dots (m) \text{ times}$

$$= n^m$$

$\therefore$ The no. of functions from $A \rightarrow B$
 $= n^m$

$$= (\text{No. of ele in B})^{\text{No. of ele in A}}$$

Now,

The total no. of Relations from $A \rightarrow B$ which are not functions

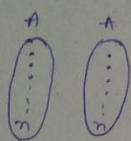
is



The no. of Relations that are not functions

$$= 2^{mn} - (n)^m$$

Now, if we are defining the function on the same set $f: A \rightarrow A$ then, The total no. of Relations that are not functions are



No. of Relations = 2^{n^2}

No. of functions = n^n

No. of Relations that are not functions = $2^{n^2} - n^n$

3. EXAMPLES ON FUNCTIONS

If there are exactly 81 functions from set A to set B then which of the following statements is not True?

a) $|A| = 4$ $|B| = 3$

b) $|A| = 2$ $|B| = 9$

c) $|A| = 1$ $|B| = 81$

d) $|A| = 9$ $|B| = 9$

No. of functions from $A \rightarrow B = (\text{No. of elements in } B)^{\text{No. of elements in } A}$

option 1 $\Rightarrow 3^4 = 81$ ✓ functions

2 $\Rightarrow 9^2 = 81$ ✓

4 $\Rightarrow 9^9 \neq 81$ $\therefore$ option 4 is false.

Which of the following is a function if domain is set of all real nos.

a) $f(x) = 1/x$ b) $f(x) = \sqrt{x}$

c) $h(x) = \pm\sqrt{x^2+1}$ d) $\phi(x) = |x|$

Domain = All Real nos.

a) $f(x) = 1/x$ If $x=0$ then the image will not be present for '0' so this is not a function.

b) $f(x) = \sqrt{x}$ for -ve no's square root is not possible (Not-function)

c) Not function because for every element there are two values $+x$ and $-x$.

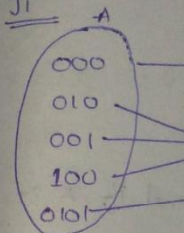
d) function.

→ consider the integers.

$f_1(s) =$ Then

$f_2(s) =$ The for which of +

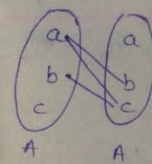
$f_1(s)$



4. EXAMPLES

Which of the f

a) $R_1 = \{(a,b), (b,c), (c,a)\}$



Not function

b) State True/False

s1) There exists

s2) The function

s3) $f(x) = \log_e x^2$

s4) The domain

s2.

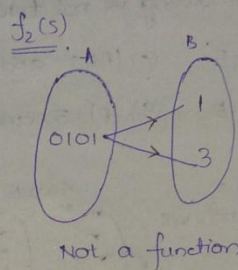
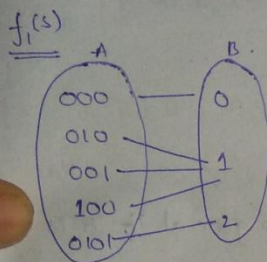
$f(x) = x$	$g(x) =$
$f(1) = 1$	$g(1) =$
$f(-1) = -1$	$g(-1) =$

→ consider the following relations from set of all bit strings to set of all integers.

$f_1(s)$ = The no. of 1's in the bit string 's'.

$f_2(s)$ = The position of a 0-bit in a bit string 's'.

which of the above relations are functions.

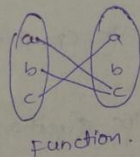
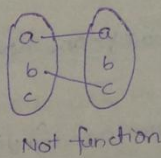
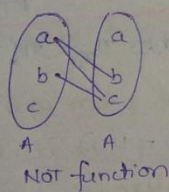


0's present at position '1' and '3', so it will be mapped to 2 elements in B.

4. EXAMPLES ON FUNCTIONS

Which of the following relations on set A is a function? $A = \{a, b, c\}$

a) $R_1 = \{(a,b)(b,c)(a,c)\}$ b) $R_2 = \{(a,a)(b,c)\}$ c) $R_3 = \{(a,c)(b,c)(c,a)\}$



b) State True/False?

S1) There exists equivalence Relation which is function. TRUE $\{(1,1)(2,2)(3,3)\}$.

S2) The functions $f(x) = x$, $g(x) = \sqrt{x^2}$ are identical. FALSE

S3) $f(x) = \log_e x^2$ and $g(x) = 2 \log_e x$ are identical. FALSE

S4) The domain of $f(x) = \frac{1}{\sqrt{|x|-x}}$ is $(-\infty, 0)$. $\Rightarrow |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$ TRUE.
 $\therefore x > 0 \Rightarrow f(x) = \frac{1}{0} \therefore f(x)$ is not defined for $x > 0$
 $\therefore \text{Domain} = (-\infty, 0)$

S2

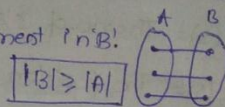
$f(x) = x \mid g(x) = \sqrt{x^2}$
 $f(1) = 1 \mid g(1) = 1$
 $f(-1) = -1 \mid g(-1) = 1$
 $(-1 \neq 1) \therefore \text{NOT IDENTICAL}$

S3
 $f(x) = \log_e x^2 \Rightarrow f(-1) = \log_e (-1)^2 = \log_e 1 = 0$
 $g(x) = 2 \log_e x = 2 \log_e (-1)$ does not exist.
 $g(-1) = \text{undefined}$

5. ONE-ONE FUNCTIONS (INJECTION)

A function 'f' from a set 'A' to set 'B' is said to be one-to-one

if no two elements in 'A' are mapped to same element in 'B'.

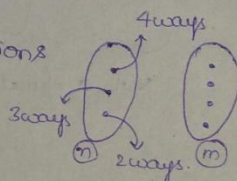


→ If there are exactly 120 one-to-one functions possible from A to B then which of the following is not true?

a) $|A|=5$ and $|B|=5$ (c) $|A|=3$ and $|B|=6$

b) $|A|=4$ and $|B|=5$ (d) $|A|=5$ and $|B|=4$

1. No. of one-to-one functions



∴ In general if 'A' is having 'n' elements and B is having 'm' elements then

the no. of one-one function from $A \rightarrow B = (m)(m-1)(m-2)(m-3) \dots (m-n+1)$

$$= {}_m P_n = \frac{(\text{No. of elements in } B)}{(\text{No. of elements in } A)}$$

if m, n are equal then the no. of one-one functions $= {}_m P_m = n!$

Now, In the question (No. of one-one functions $= {}_m P_n = 120$)

→ op1: $|A|=5$ $|B|=5 = {}_5 P_5 = 5! = 120$. TRUE

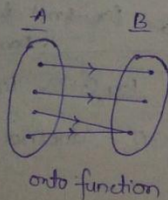
→ op2: $|A|=4$ $|B|=5 = {}_5 P_4 = 5 \times 4 \times 3 \times 2 = 120$. TRUE

→ op3: $|A|=3$ and $|B|=6 \Rightarrow {}_6 P_3 = 6 \times 5 \times 4 = 120$.

→ op4: $|A|=5$ $|B|=4$ (${}_{4 P 5}$) - Not possible

6. ONTO FUNCTIONS

A function $f: A \rightarrow B$ is said to be onto if each element of 'B' is mapped by atleast one element of 'A'. i.e. Range of $f = B$.



The condition for a function to be onto is $|B| \leq |A|$

→ If this doesn't satisfy then func. is not onto.

→ we cannot say a func. is onto if it satisfies the above condition.

If $|A| = |B|$

7. EXAMPLE

If $|A| = m$ from A to B n^m

Ex: If $|A| =$

$|A| = 6 \Rightarrow$

$|B| = 3$

Ex: If $|A| = n$

$\Rightarrow 2^n$

8. EXAMPLES

In how many so that every project is as

sol:

If $|A| = |B|$ then the no. of onto functions = $n!$.

1. EXAMPLES ON ONTO FUNCTIONS - 1

If $|A| = m$ and $|B| = n$, ($m > n$) then the no. of onto functions possible from A to B is.

$$n^m - n_c (n-1)^m + n_c^2 (n-2)^m - n_c^3 (n-3)^m + \dots + (-1)^n n_c^{m-1} (1)^m$$

Ex: If $|A| = 6$, $|B| = 3$ then the no. of onto functions from A to B is —?

$$|A| = 6 \Rightarrow m = 6$$

$$|B| = 3 \Rightarrow n = 3$$

$$\Rightarrow 3^6 - 3_c (2)^6 + 3_c^2 (1)^6 - 3_c^3 (0)^6 = 729 - 3 \times 64 + 3 \times 1 = 540$$

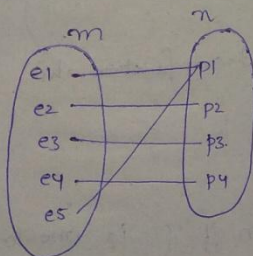
Ex: If $|A| = n$, $|B| = 2$ ($n > 2$) then the no. of onto functions from A to B?

$$\Rightarrow 2^n - 2_c (1)^n = 2^n - 2$$

2. EXAMPLES ON ONTO FUNCTIONS - 2

In how many ways we can assign 5 employees to 4 projects so that every employee is assigned to only one project and every project is assigned to at least one employee?

sol:



$$m = 5 \quad n = 4$$

$$\text{Required ways} = 4^5 - 4_c (3)^5 + 4_c^2 (2)^5 - 4_c^3 (1)^5 + 0$$

$$= 1024 - (243 \times 4) + 6(32) + (-4) = 240$$

240 ways.

9. EXAMPLES ON ONTO FUNCTIONS - 3

Consider the following functions on set of all integers $f(x) = x^2$, $g(x) = x^3$ and $h(x) = \lceil x/2 \rceil$ which of the following is TRUE?

- R) s1) f_1 is one-one (FALSE) s4) g is onto (FALSE)
 T) s2) f is onto (FALSE) s5) h is one-one (TRUE)
 S3) g is one-one (TRUE) s6) h is - onto

1)

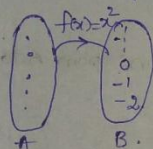
$$f(x) = x^2$$

$$f(1) = 1$$

$$f(-1) = 1$$

Two elements are mapped to same element so NOT ONE-ONE

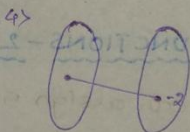
2) Set of Integers = $\{-\infty \dots 0 \dots \infty\}$.



square of a number cannot be Negative but so Negative nos in B cannot be mapped with any element in A not one-one.

$$3) g(x) = x^3$$

Not one-one function



Not pre Image for (-2) $\therefore$ Not onto

$$5) h(x) = \lceil x/2 \rceil$$

$$h(1) = 1$$

$$h(2) = 1$$

Not one-one function. and onto function also.

10. BIJECTION

A function $f: A \rightarrow B$ is called a Bijection if 'f' is one-to-one as well as onto.

$\rightarrow$ If 'A' and 'B' are finite sets then Bijection from 'A' to 'B' is possible

$$\text{if } |A| = |B|$$

$\rightarrow$ If $|A| = |B| = n$ then No. of Bijections possible from A to B is $n!$

11. EXAM

Let $A = \mathbb{R}$

$$f(x) = \frac{x-2}{x-3}$$

a) f is one

b) f is onto

Let $f(x)$

$$= \frac{a-2}{a-3}$$

$$= (a-2)$$

$$= a-2$$

$$\Rightarrow a$$

12. INVERSE

Let $f: A \rightarrow$

is called

Theorem:

a) which of

a) $f(x)$

one-one

$$f: A \rightarrow B$$

$$|A| \leq |B|$$

onto function

$$f: A \rightarrow B$$

$$|A| \geq |B|$$

$$|A| = |B|$$

Bijection function condition.

65

five 1/8

(A)

II. EXAMPLE ON BIJECTION

Let $A = \mathbb{R} - \{3\}$ and $B = \mathbb{R} - \{1\}$. A function $f: A \rightarrow B$ is defined by

$$f(x) = \frac{x-2}{x-3} \text{ which of the following is true?}$$

- a) f is one-one but not onto (X) c) f is bijection (X)
b) f is onto but not one-one (d) f is neither one-one or onto

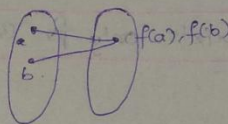
1) Let $f(a) = f(b)$

$$= \frac{a-2}{a-3} = \frac{b-2}{b-3}$$

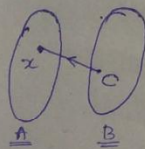
$$= (a-2)(b-3) = (b-2)(a-3)$$

$$= ab - 3a - 2b + 6 = ab - 3b - 2a + 6$$

$$\Rightarrow a = b \text{ Not one to one}$$



2)



$$f(x) = c$$

$$\Rightarrow \frac{x-2}{x-3} = c$$

$$\Rightarrow x-2 = cx-3c$$

$$\Rightarrow x(1-c) = -3c+2$$

$$\Rightarrow x = \frac{2-3c}{1-c}$$

for every element 'c' we can find an element 'x' in 'A'.

12. INVERSE OF A FUNCTION

possible

Let $f: A \rightarrow B$, if the inverse relation $f^{-1}: B \rightarrow A$ is a function then it is called inverse of function 'f' and it is denoted by $f^{-1}: B \rightarrow A$

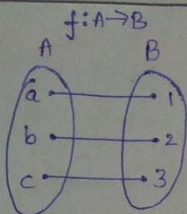
Theorem: Inverse of $f: A \rightarrow B$ exists iff 'f' is a bijection

Q) which of the following functions have inverse defined on their Ranges?

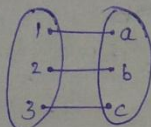
- a) $f(x) = x^2$ b) $f(x) = x^3$

80)

function.

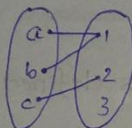


$f^{-1}: B \rightarrow A$



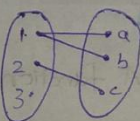
function $\Rightarrow f^{-1}$ exists.

$f: A \rightarrow B$



function

$f^{-1}: B \rightarrow A$



Not function

for a function to have Inverse it should be one-one function and onto function.

$\therefore$ Inverse exists only for Bijection functions.

a) $f(x) = x^2$

$\left. \begin{matrix} f(1) = 1 \\ f(-1) = 1 \end{matrix} \right\}$ Not one-one $\Rightarrow$ Not Bijection.

b) $f(x) = x^3 \rightarrow$ one-one, onto, Bijection $\Rightarrow$ Inverse exists.